TTN 1

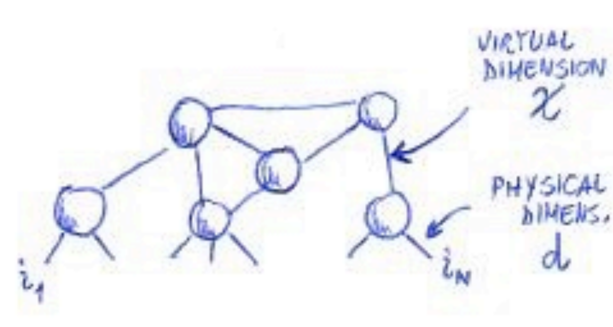
Lecture on TREE TENSOR NETWORKS

LATTICE QUANTUM SYSTEM } | # SITES N | LOCAL DIMENSION d | PHYSICAL DIMENSION/LATTICE GEOMETRY 1D, 2D... | OTHER
↳ STATISTICS
SPIN/BOSON VS FERM
↳ INTERACTIONS

QUANTUM MANY-BODY (PURE) STATE

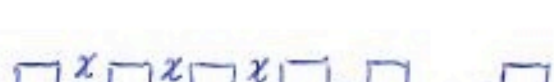
$$|\Psi\rangle = \sum_{i_1 \dots i_N = 1}^{d} T_{i_1 \dots i_N} \underbrace{|i_1 \dots i_N\rangle}_{\text{(REAL SPACE) PRODUCT BASIS}}$$

≈ TAILORED VARIATIONAL WAVEFUNCTION ANSATZ



FOR 1D ➤ MATRIX PRODUCT STATES ARE VERY SUCCESSFUL
↳ OBC FINITE SYSTEM
↳ INFINITE SYSTEM

ÖSTLUND, ROMMER; PRL 75, 3537 (1995)
SCHOLLWÖCK; ANN. PHYS 326, 96 (2011)

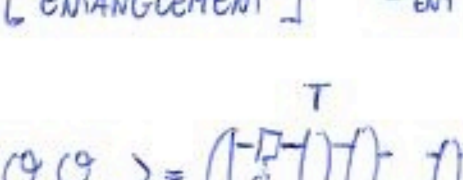


Several interesting properties

① Loopless (EXCEPT FINITE PBC)
→ CANONICAL FORM CAN BE REACHED WITH SINGLE-TENSOR STD. LINEAR ALGEBRA
→ SIMPLE EIGENPROBLEMS (NOT GENERALIZED)

② Area-Law
FOR NONCRITICAL 1D QUANTUM STATES $S \propto L^0$
[BIPARTITION ENTANGLEMENT] $S_{ENT} \leq \log \chi \rightarrow \chi \propto L^0$

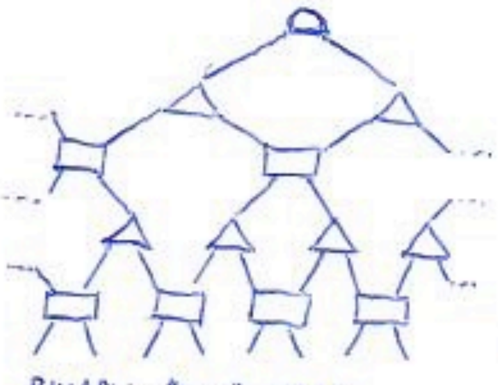
③ NON-CRITICAL Correlations

$\langle O_j O_{j+\ell} \rangle =$  $= Q_L T^{\ell} Q_R = \sum_q c_q e^{-\ell/\xi_q}$

WHAT IF I WANT TO REPRESENT A CRITICAL THEORY ?

$$S \approx \frac{c}{6} \log L \quad \text{BUT} \quad S \leq \log \chi \Rightarrow \underbrace{\chi \geq b L^{c/6}}_{\text{MUST SCALE!}}$$

NOT A DEALBREAKER BUT STILL



BINARY "FULL" MERA

MERA

VIDAL; PRL 99, 220405 (2007)

MULTISCALE ENTANGLEMENT RENORMALIZATION ANSATZ

DISENTANGLERS

UNITARIES $\mathcal{U}$

$\mathcal{U}^\dagger \mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = \mathbb{1}$

ISOMETRIES

ISOMETRY V

$V^\dagger V = \mathbb{1}$

- ✓ REPRESENT CRITICAL STATES AF FINITE (STATIC) χ

$$\langle O_J \, O_{J+\ell} \rangle = \dots = \sum_q c_q \, \ell^{\eta_q}$$

MONTANGERO [redacted] et al. PRB 80, 113103 (2009)

- ! INTERESTING CONNECTIONS WITH HOLOGRAPHY

MERA $\longleftrightarrow$ AdS/CFT

SWINGLE; PRD 86, 065007 (2012)

- ✗ LOOPY
  - → CANONICAL FORM CAN NOT BE ESTABLISHED
  - → A NON-FLEXIBLE CANONICAL FORM HAS TO BE ENFORCED BY UNITARY DISENTANGLERS
    - ↳ CONSTRAINED PROBLEM OF LINEAR ALGEBRA
  - → GENERALIZED EIGENVALUE PROBLEMS

ALSO PEPS, [redacted] PBCMPS, LPTN SUFFER FROM THIS DRAWBACK

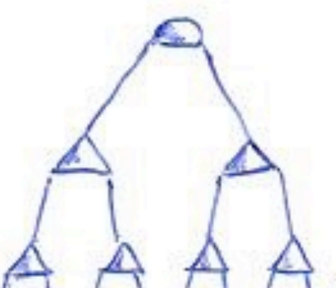
- ✗ FINALLY, THE SCALING OF ALGORITHM IS χ^8 OR WORSE

FOR COMPARISON $\left[\text{MPS}_{\text{SHORT RANGE}} \ \chi^3 d^2 \ ; \ \text{MPS}_{\text{LONG RANGE}} \ \min(\chi^4 d^2, N^2 \chi^3 d^2) \right]$

AN INTERESTING OPTION TO ADDRESS THESE PROBLEMS

$\mathcal{U} \rightarrow | \ | \ \mathbb{1}$

REMOVE ALL UNITARIES



TREE TENSOR NETWORK

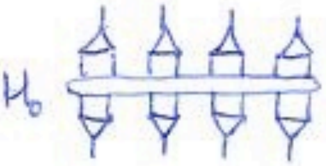
← BINARY TTN ANSATZ

TTN CONCEPTUAL SIGNIFICANCE

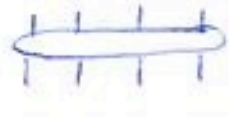
→ LIKE AN MPS, IS THE TENSOR NETWORK ANSATZ RELATED TO DMRG

→ THE TTN, IS THE TENSOR NETWORK ANSATZ RELATED TO REAL-SPACE RG-FLOW

LATTICE HAMILTONIAN (1D) → H_0



⇒ MAPPED HAMILTONIAN



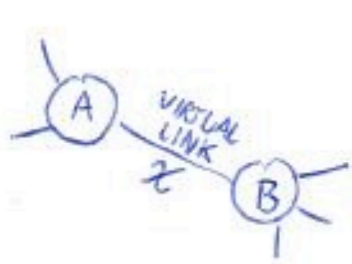
AFTER 1 RG STEP

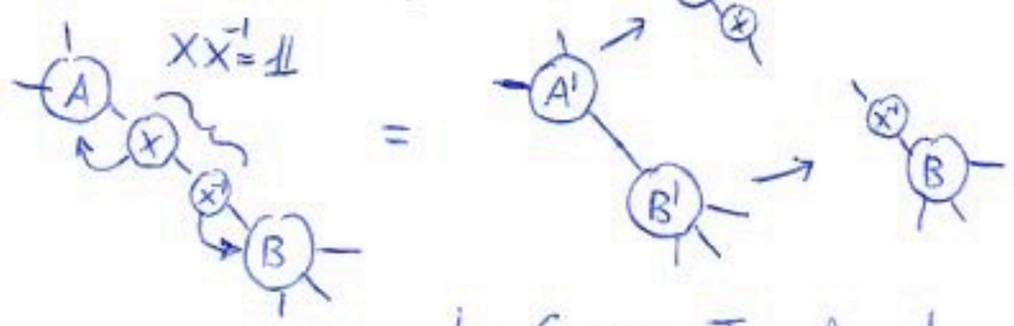
TTN - Properties:

(1) Loopless → FLEXIBLE CANONICAL FORM
→ ANY TENSOR CAN BE TURNED INTO GAUGE CENTER VIA SINGLE-TENSOR LINEAR ALGEBRA

What does that mean?

[A] TENSOR NETWORK GAUGE TRANSFORMATION
(X INVERTIBLE MATRIX)



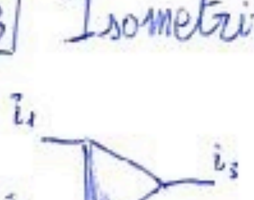


→ SAME STATE
→ SAME NETWORK GEOMETRY

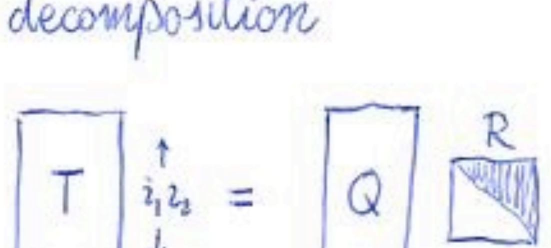
↳ Gauge Transformation

[B] Isometrization via QR decomposition

{ "THIN" QR FACTORIZATION



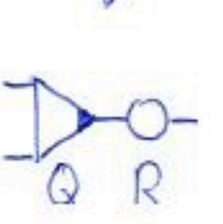
WRITTEN AS A MATRIX
$(i_1, i_2) \times i_3$
$\chi^2 \times \chi$
ROWS COLUMNS



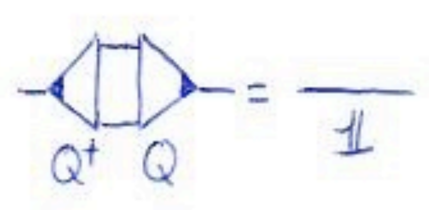
$Q^\dagger Q = \mathbb{1}_{\chi \times \chi}$

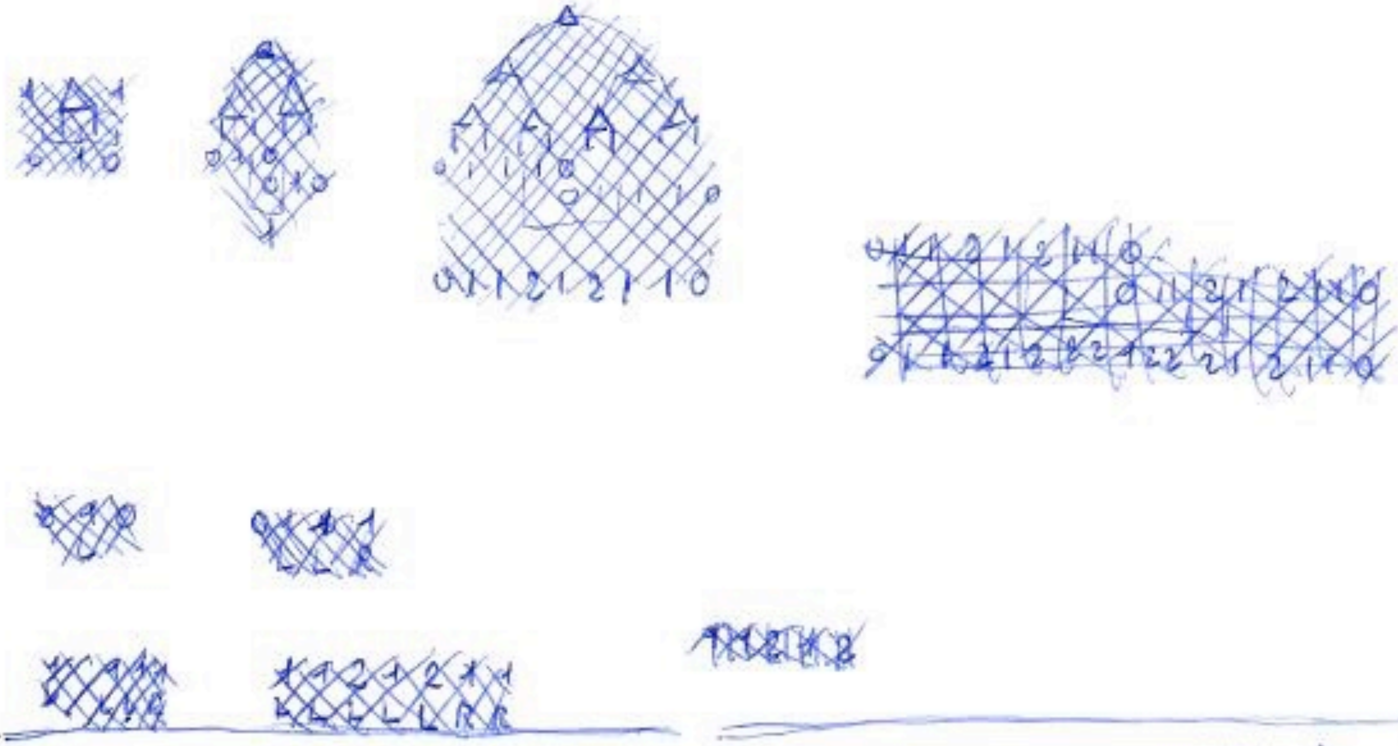
R UPPER TRIANGULAR $\chi \times \chi$

THE R CAN BE ABSORBED IN THE ADJACENT TENSOR
↳ GAUGE TRANSFORMATION

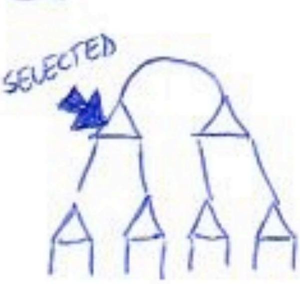


WHERE





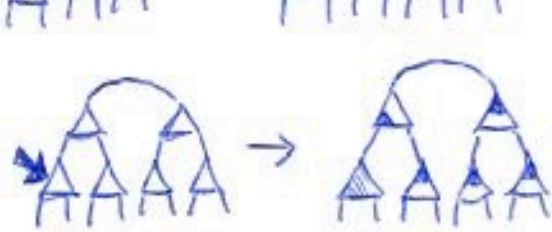
C Establishing the CENTRAL GAUGE (TOWARDS ANY NODE)



→ STEP 1: DETERMINE NETWORK-GEOMETRY DISTANCES FROM THE TARGET NODE

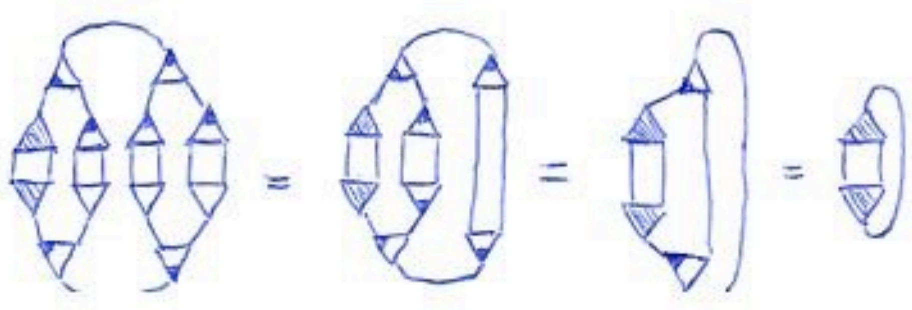
→ STEP 2: FROM HIGHER TO LOWER DISTANCES REPEAT
↳ QR DECOMPOSITION TOWARDS LOWER DISTANCE
↳ ABSORB (R) MATRIX INTO LOWER DISTANCE TENSOR

IT'S NOT ALWAYS UPWARD



Huge Benefit

$\langle\Psi|\Psi\rangle =$

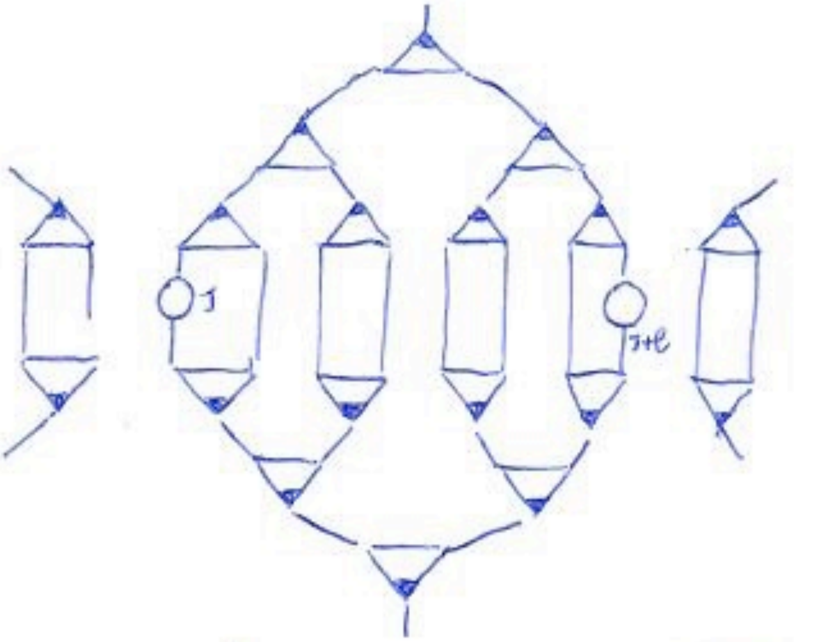


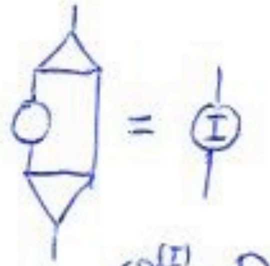
FROBENIUS NORM SQUARE OF T'

② CRITICALITY AT FINITE χ (EVEN THOUGH ENTANGLEMENT ENTROPY IS FRACTAL IN THE PARTITION)

↳ WE ASSUME A <u>BULK</u> OF HOMOGENEOUS TENSORS [GAUGED <u>UP</u>]

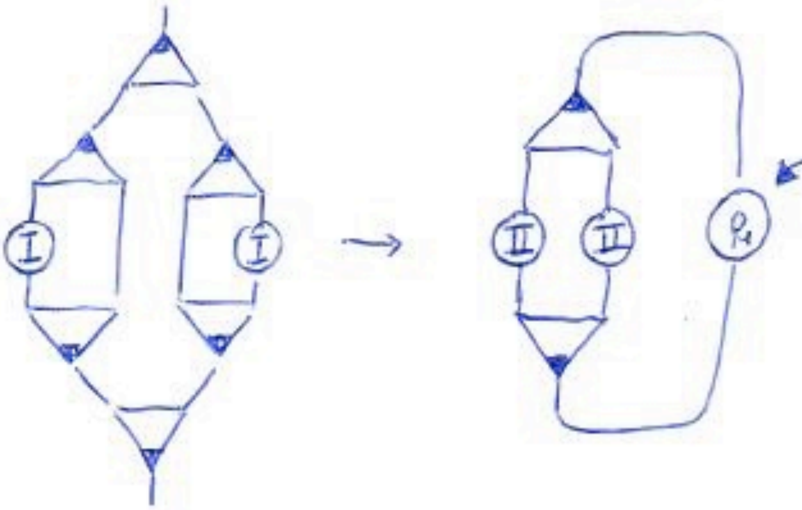
↳ LET US CALCULATE $\langle O_J O_{J+\ell} \rangle$ WITH $J = 2^q + 1$ AND $\ell = 2^K - 1$





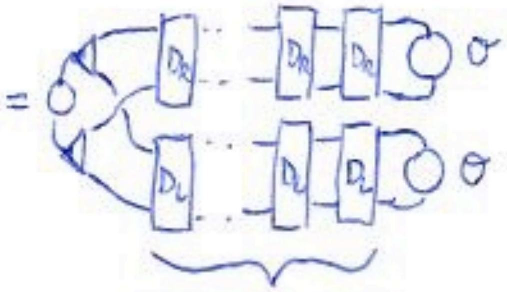
COMPLETELY POSITIVE UNITAL MAP

$O_q^{(I)} = D_L(O)$



ALL THE REST OF THE TENSOR NETWORK CONTRACTS INTO A DENSITY MATRIX $\chi \times \chi$

$$\langle O_J \, O_{J+2^K-1} \rangle = \mathrm{Tr}\left[\rho_1 \, T \left(D_L^K(O) \otimes D_R^K(O) \right) T^\dagger \right]$$



$$= \sum c_q \lambda_q^K = \sum c_q e^{-K \eta_q'}$$

$$= \sum c_q \left(e^{\log_2(\ell+1)} \right)^{-\eta_q'} = \sum c_q (\ell + 1)^{-\eta_q}$$

FOR AVERAGES

POWER-LAW CORRELATIONS

③ Algorithms that scale PRETTY WELL χ^4 {SINGLE TENSOR UPDATE

(SHORT RANGE INTERACTION $L\chi^4$; LONG RANGE INTERACTION $L^2\chi^4$; THREE BODY INTERACTION $\leq \chi^6$)

The DMRG-LIKE algorithm FOR TTN

GERSTER et al PRB 90, 125154 (2014)

(TASK) GROUND STATE OF AN HAMILTONIAN H:

MINIMIZE $\langle\Psi|H|\Psi\rangle$ WITH CONSTRAINT $\langle\Psi|\Psi\rangle = 1$

Lagrangian $\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle = \mathcal{L}$

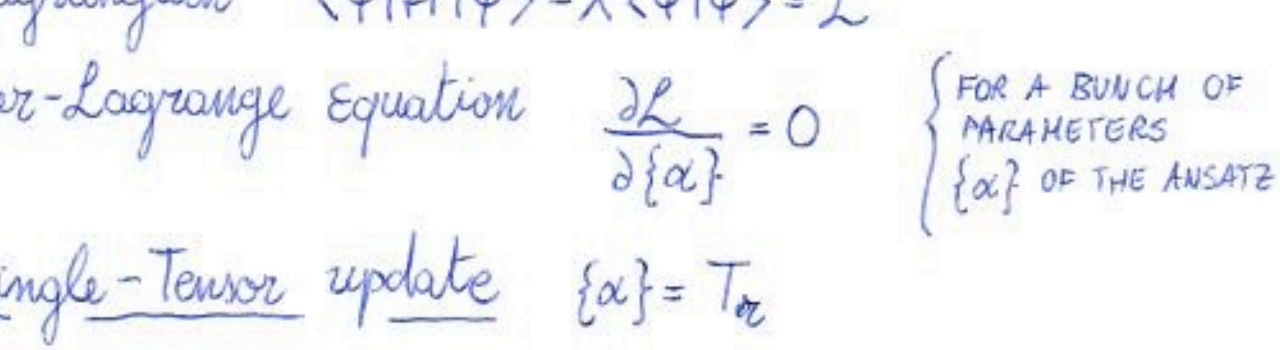
Euler-Lagrange Equation $\dfrac{\partial \mathcal{L}}{\partial\{\alpha\}} = 0$ {FOR A BUNCH OF PARAMETERS $\{\alpha\}$ OF THE ANSATZ

(RULE) Single-Tensor update $\{\alpha\} = T_{\tau}$

↳ CONVERT THE TTN IN CENTRAL GAUGE TOWARDS NODE τ

↳ FIX ALL TENSORS EXCEPT T_{τ} AND VARIATE THAT

$\dfrac{\partial \mathcal{L}}{\partial T_{\tau}^{*}} =$

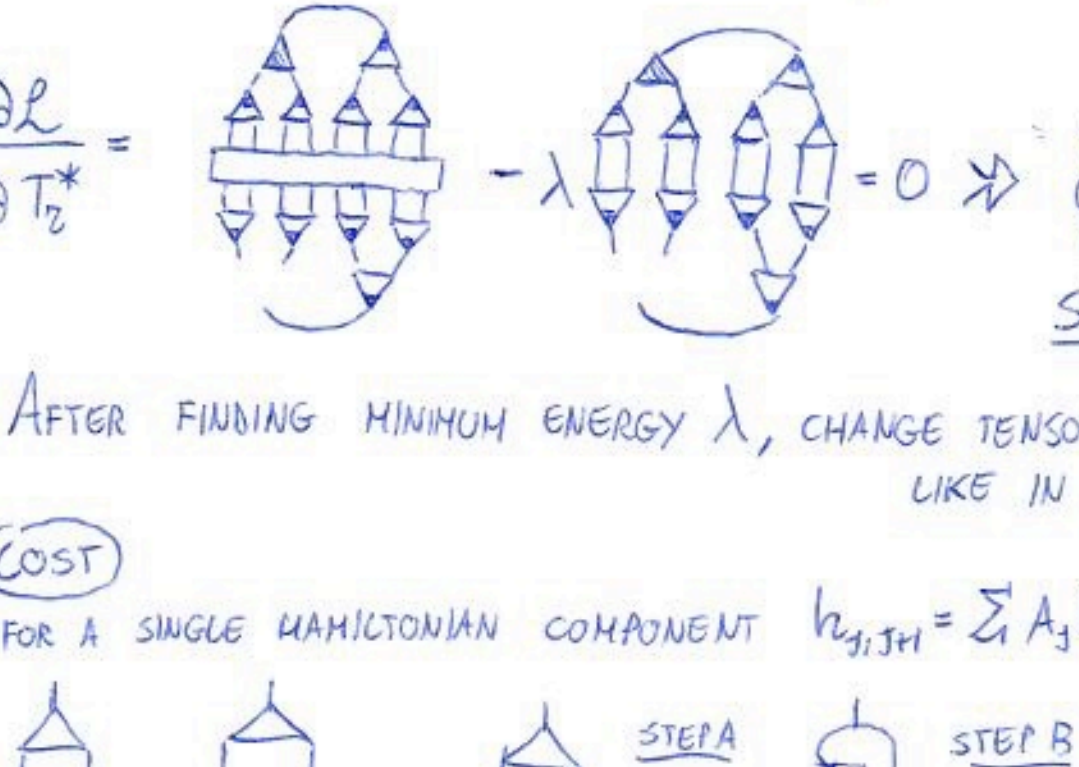


SIMPLE EIGENPROBLEM

AFTER FINDING MINIMUM ENERGY λ, CHANGE TENSOR AN REPEAT, JUST LIKE IN MEAN-FIELD.

(COST)

FOR A SINGLE HAMILTONIAN COMPONENT $h_{j,j+1} = \sum_i A_j B_{j+1}$



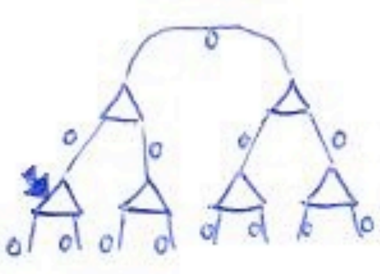
ALSO OTHER SUBROUTINES COST THE SAME

The TIME-DEPENT VARIATIONAL ALGORITHM (TDVP) for TTN

Evolve the TTN state $|\Psi_{TTN}\rangle$ according to Schrödinger equation $\frac{d}{dt}|\Psi\rangle = -iH|\Psi\rangle$

MPS-TDVP	HAEGEMANN et al PRL 107, 070601 (2011)
TTN-TDVP	KOHN et al PRA 101, 023617 (2020)

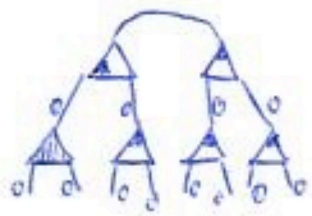
(RULE) SINGLE TENSOR UPDATE (CHEAPEST χ^4)



WRITE THE "INTERNAL TIME" FOR EACH SPACE, {PHYSICAL AND VIRTUAL

↳ OUR GOAL IS TO EVOLVE TO "+1" (ONE TIME-STEP FORWARD) EVERYWHERE

GAUGE AND EFFECTIVE HAMILTONIAN



;

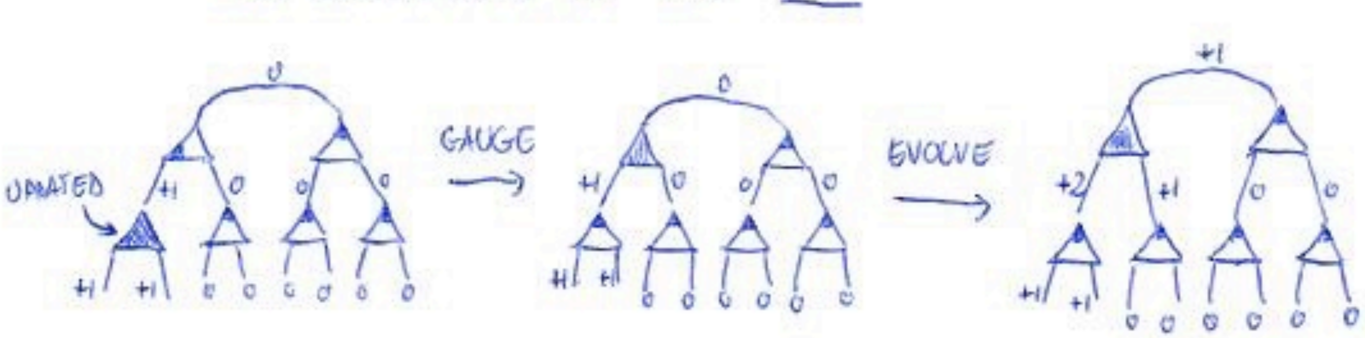
$H_{eff} = \langle\Psi|H|\Psi\rangle$

INTEGRATE evolve T using H_{eff} $\frac{d}{dt}|T) = -iH_{eff}|T)$

$|T(0)) \longrightarrow |T(+dt))$

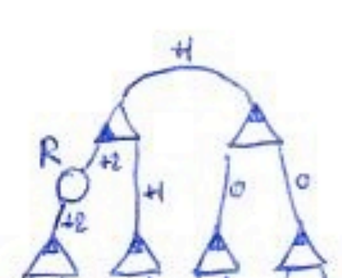
ANY SCHRÖDINGER INTEGRATOR, FOR EXAMPLE RUNGE-KUTTA

↳ SUBSTITUTE IN THE TTN



THE +2 LINK EVOLVED "TOO MUCH", WE NEED TO SEND IT BACKWARDS

GAUGE →



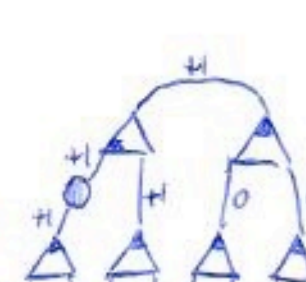
→ CALCULATE

$= \langle\Psi|H|\Psi\rangle$

EVOLVE R WITH K_{eff}

BY $(-dt)$ ← BACKWARDS IN TIME

→

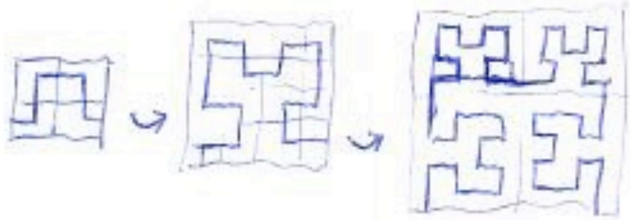


CONTINUE UNTIL ALL "+1"

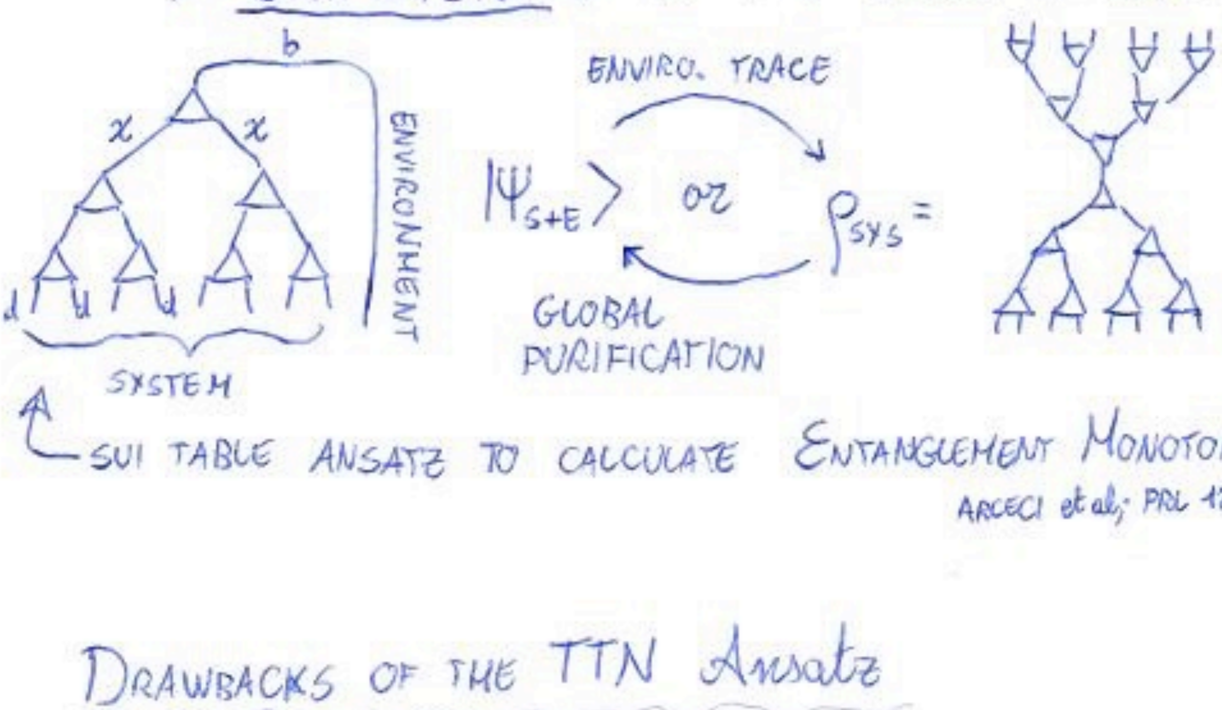
(4) Geometries SUITABLE for TTN

- ↳ 1D FINITE OBC ✓ OFTEN COMPETITIVE WITH DMRG
- ↳ 1D FINITE PBC ✓ EXCELLENT!
- ↳ 1D INFINITE — WELL... THAT'S WILSON'S RG, WHICH WORKS SINCE 60 years
- ↳ 2D FINITE ?! IT WORKS WELL SOMETIMES (LATTICE QED)
  - ↳ HINDERED BY RVB (PEPS-LIKE STATES)

BY THE WAY
FOR 2D SQUARE LATTICE WITH 3-LEG TENSOR TTN } MAP THE 2D LATTICE INTO 1D WITH HILBERT CURVE



↳ OPEN SYSTEM: THE TREE TENSOR OPERATOR ANSATZ



↳ SUITABLE ANSATZ TO CALCULATE ENTANGLEMENT MONOTONES
ARCECI et al; PRL 128, 040501 (2022)

DRAWBACKS OF THE TTN Ansatz

☹ TRANSLATIONAL INVARIANCE IS BROKEN (ALSO FOR MERA)
↳ IMPOSSIBLE TO SIMULTANEOUSLY ENFORCE SCALE AND TRANSLATION ON A LATTICE

☹ Entanglement CLUSTERING due TO "UNFORTUNATE" PARTITIONS
FERRIS; PRB 87, 125139 (2013)