Simulating quantum fields: from discretuum to continuum

arXiv:2109.14214; arXiv:2107.13834; arXiv:1901.06124

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Standard Model of

FUNDAMENTAL PARTICLES AND INTERACTIONS

The Standard Model summarizes the current knowledge in Particle Physics. It is the quantum theory that includes the theory of strong interactions (quantum chromodynamics or QCD) and the unified theory of weak and electromagnetic interactions (electroweak). Gravity is included on this chart because it is one of the fundamental interactions even though not part of the "Standard Model."

FERMIONS

matter constituents
spin = 1/2, 3/2, 5/2, ...

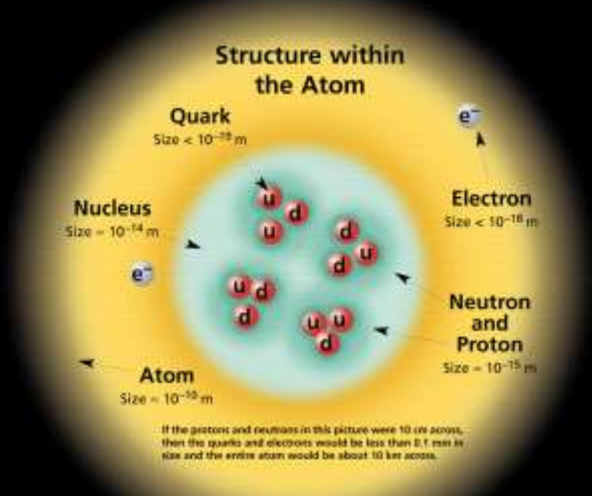
Leptons spin = 1/2		
Flavor	Mass GeV/c^2	Electric charge
ν_e electron neutrino	$<1\times10^{-8}$	0
e electron	0.000511	−1
ν_μ muon neutrino	<0.0002	0
μ muon	0.106	−1
ν_τ tau neutrino	<0.02	0
τ tau	1.7771	−1

Quarks spin = 1/2		
Flavor	Approx. Mass GeV/c^2	Electric charge
u up	0.003	2/3
d down	0.006	−1/3
c charm	1.3	2/3
s strange	0.1	−1/3
t top	175	2/3
b bottom	4.3	−1/3

Spin is the intrinsic angular momentum of particles. Spin is given in units of $\hbar$, which is the quantum unit of angular momentum, where $\hbar = h/2\pi = 6.58\times10^{-25}$ GeV s $= 1.05\times10^{-34}$ J s.

Electric charges are given in units of the proton's charge. In SI units the electric charge of the proton is 1.60×10^{-19} coulombs.

The **energy** unit of particle physics is the electronvolt (eV), the energy gained by one electron in crossing a potential difference of one volt. **Masses** are given in GeV/c^2 (remember $E = mc^2$), where 1 GeV $= 10^9$ eV $= 1.60\times10^{-10}$ joule. The mass of the proton is 0.938 GeV/c^2 $= 1.67\times10^{-27}$ kg.



BOSONS

force carriers
spin = 0, 1, 2, ...

Unified Electroweak spin = 1		
Name	Mass GeV/c^2	Electric charge
γ photon	0	0
W^-	80.4	−1
W^+	80.4	+1
Z^0	91.187	0

Strong (color) spin = 1		
Name	Mass GeV/c^2	Electric charge
g gluon	0	0

Color Charge

Each quark carries one of three types of "strong charge," also called "color charge." These charges have nothing to do with the colors of visible light. There are eight possible types of color charge for gluons. Just as electrically-charged particles interact by exchanging photons, in strong interactions color-charged particles interact by exchanging gluons. Leptons, photons, and ***W*** and ***Z*** bosons have no strong interactions and hence no color charge.

Quarks Confined in Mesons and Baryons

One cannot isolate quarks and gluons; they are confined in color-neutral particles called **hadrons**. This confinement (binding) results from multiple exchanges of gluons among the color-charged constituents. As color-charged particles (quarks and gluons) move apart, the energy in the color-force field between them increases. This energy eventually is converted into additional quark-antiquark pairs (see figure below). The quarks and antiquarks then combine into hadrons; these are the particles seen to emerge. Two types of hadrons have been observed in nature: **mesons** $q\bar{q}$ and **baryons** qqq.

Residual Strong Interaction

The strong binding of color-neutral protons and neutrons to form nuclei is due to residual strong interactions between their color-charged constituents. It is similar to the residual electrical interaction that binds electrically neutral atoms to form molecules. It can also be viewed as the exchange of mesons between the hadrons.

PROPERTIES OF THE INTERACTIONS

Property \ Interaction	Gravitational	Weak (Electroweak)	Electromagnetic	Strong: Fundamental	Strong: Residual
Acts on:	Mass – Energy	Flavor	Electric Charge	Color Charge	See Residual Strong Interaction Note
Particles experiencing:	All	Quarks, Leptons	Electrically charged	Quarks, Gluons	Hadrons
Particles mediating:	Graviton (not yet observed)	W^+ W^- Z^0	γ	Gluons	Mesons
Strength relative to electromag for two u quarks at: 10^{-18} m	10^{-41}	0.8	1	25	Not applicable to quarks
Strength relative to electromag for two u quarks at: 3×10^{-17} m	10^{-41}	10^{-4}	1	60	
for two protons in nucleus	10^{-36}	10^{-7}	1	Not applicable to hadrons	20

Baryons qqq and Antibaryons $\bar{q}\bar{q}\bar{q}$ — Baryons are fermionic hadrons. There are about 120 types of baryons.					
Symbol	Name	Quark content	Electric charge	Mass GeV/c^2	Spin
p	proton	uud	1	0.938	1/2
$\bar{p}$	anti-proton	$\bar{u}\bar{u}\bar{d}$	−1	0.938	1/2
n	neutron	udd	0	0.940	1/2
Λ	lambda	uds	0	1.116	1/2
Ω^-	omega	sss	−1	1.672	3/2

Mesons $q\bar{q}$ — Mesons are bosonic hadrons. There are about 140 types of mesons.					
Symbol	Name	Quark content	Electric charge	Mass GeV/c^2	Spin
π^+	pion	$u\bar{d}$	+1	0.140	0
K^-	kaon	$s\bar{u}$	−1	0.494	0
ρ^+	rho	$u\bar{d}$	+1	0.770	1
B^0	B-zero	$d\bar{b}$	0	5.279	0
η_c	eta-c	$c\bar{c}$	0	2.980	0

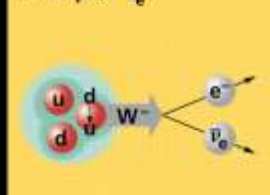
Matter and Antimatter

For every particle type there is a corresponding antiparticle type, denoted by a bar over the particle symbol (unless + or – charge is shown). Particle and antiparticle have identical mass and spin but opposite charges. Some electrically neutral bosons (e.g., Z^0, γ, and $\eta_c = c\bar{c}$, but not $K^0 = d\bar{s}$) are their own antiparticles.

Figures

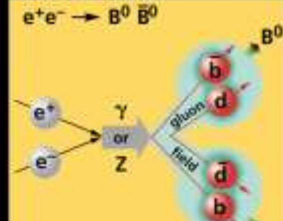
These diagrams are an artist's conception of physical processes. They are **not** exact and have ***no*** meaningful scale. Green shaded areas represent the cloud of gluons or the gluon field, and red lines the quark paths.

$n \rightarrow p\, e^-\, \bar{\nu}_e$



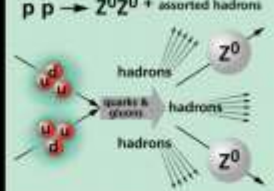
A neutron decays to a proton, an electron, and an antineutrino via a virtual (mediating) W boson. This is neutron β decay.

$e^+e^- \rightarrow B^0\, \bar{B}^0$



An electron and positron (antielectron) colliding at high energy can annihilate to produce B^0 and $\bar{B}^0$ mesons via a virtual Z boson or a virtual photon.

$p\, p \rightarrow Z^0 Z^0$ + assorted hadrons



Two protons colliding at high energy can produce various hadrons plus very high mass particles such as Z bosons. Events such as this one are rare but can yield vital clues to the structure of matter.

The Particle Adventure

Visit the award-winning web feature *The Particle Adventure* at
http://ParticleAdventure.org

This chart has been made possible by the generous support of:
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American Physical Society, Division of Particles and Fields
BURLE INDUSTRIES, INC.

http://CPEPweb.org

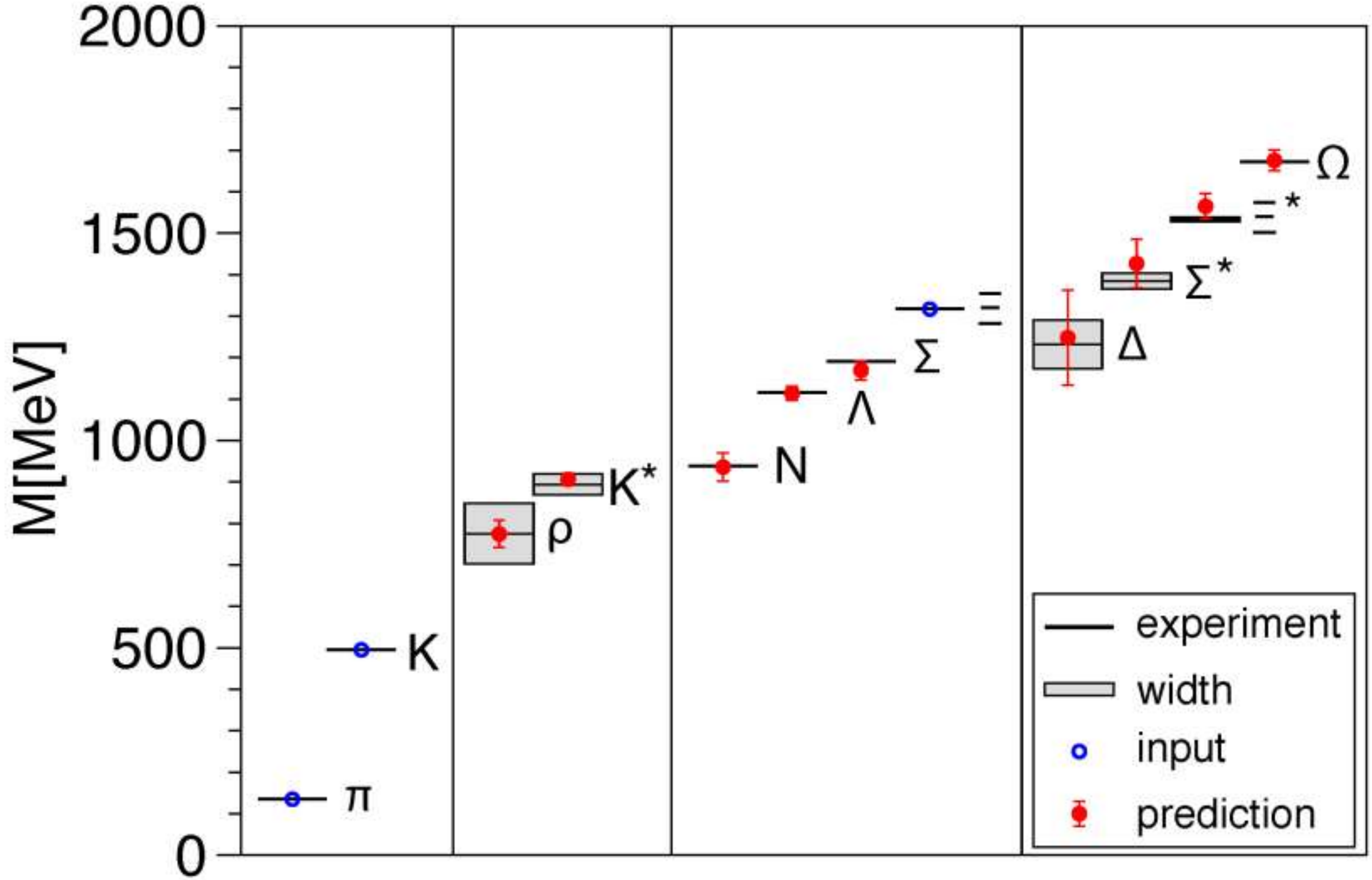
$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) + \bar{\psi}(i\not{D} - m)\psi + \cdots$$

Prior work (classical simulation)

Mathematics: constructive QFT, algebraic QFT, Segal QFT, SLE, ... [1]

Physics: “traditional QFT” [2], lattice QFT [3], tensor networks [4,5]

[1] M. R. Douglas, Proc. Symp. Pure Math., 85 (2012)
[2] see, e.g., M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory* (1995)
[3] see, e.g., M. Creutz, *Quarks, gluons and lattices* (1985)
[4] F. Verstraete and J. I. Cirac, Phys. Rev. Lett. 104, 190405 (2010)
[5] J. Haegeman, TJO, H. Verschelde, and F. Verstraete, Phys. Rev. Lett. 110, 100402 (2013)



Z. Fodor, C. Hoelbling, Rev. Mod. Phys. **84**, 449 (2012)

Statics of QFT can be ”easily” simulated (classically)

Dynamics of QFT?

Classically hard!

E.g., sign problem

Quantum information era:

Quantum simulation evades the sign problem

Prior work (quantum simulation)

- Gauge theories [1]
- Gaugelike theories (e.g. link models) [2]
- ϕ^4 theory [3]
- Fermionic theories and beyond [4,5]

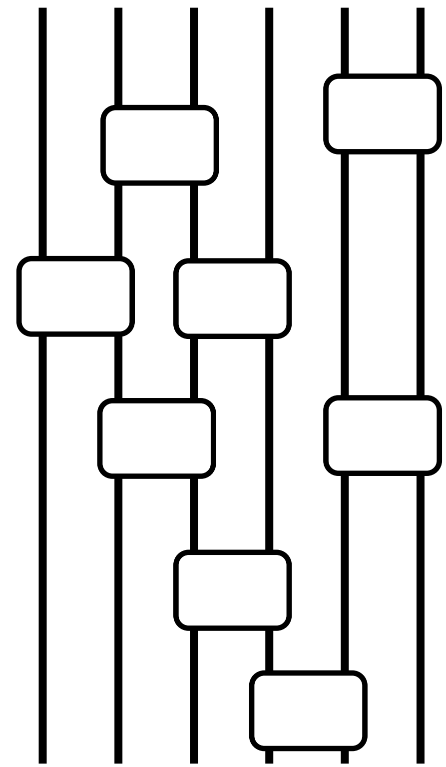
[1] T. Byrnes and Y. Yamamoto, Phys. Rev. A 73, 022328 (2006)
[2] M. C. Bañuls et al, Eur. Phys. J. D 74, 165 (2020)
[3] S. P. Jordan, K. S. M. Lee, J. Preskill, Science 336, 1130 (2012)
[4] S. P. Jordan, K. S. M. Lee, J. Preskill, arXiv:1404.7115 (2014)
[5] J. Preskill, LATTICE2018. arXiv:1811.10085 (2018)

What does it even mean to **"simulate" a QFT?**

- Identify masses of particles?
- Reproduce some correlators?
- Produce approx. to ground state?
- Approximate the S matrix?
- Simulate Poincare/conformal group with error bounds

Poincare group approx.:

$$e^{-\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}} \sim$$



Simulate QFT with error bounds ~

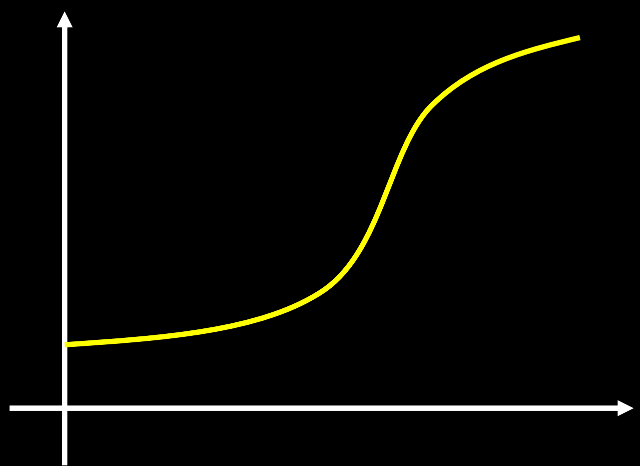
mathematically rigourous QFT

Notoriously difficult!

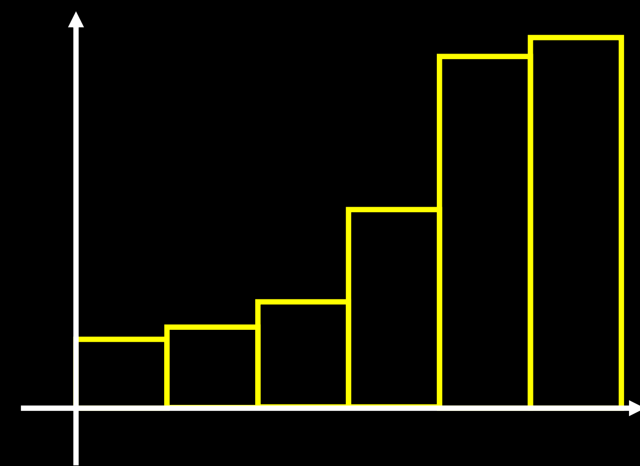
... but not for the reasons you might think

https://www.claymath.org/millennium-problems/yang%E2%80%93mills-and-mass-gap

KINEMATICS



$\overset{?}{\sim}$

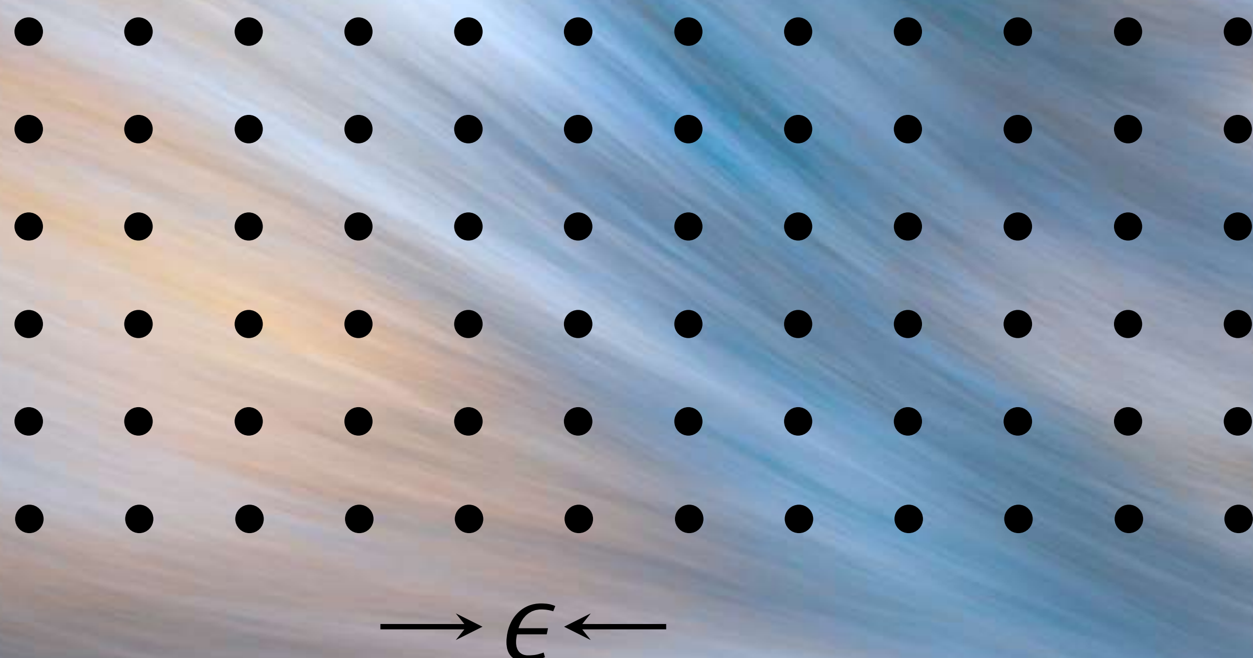


$$|\psi_{\text{Field}}\rangle \sim |\psi_{\text{Discrete}}\rangle$$

Wilsonian QFT: adjust a regulated theory by increasing cutoff while preserving low-energy predictions

K.G. Wilson, *The renormalization group: critical phenomena and the Kondo problem*, Rev. Mod. Phys. 47, 4, 773 (1975)

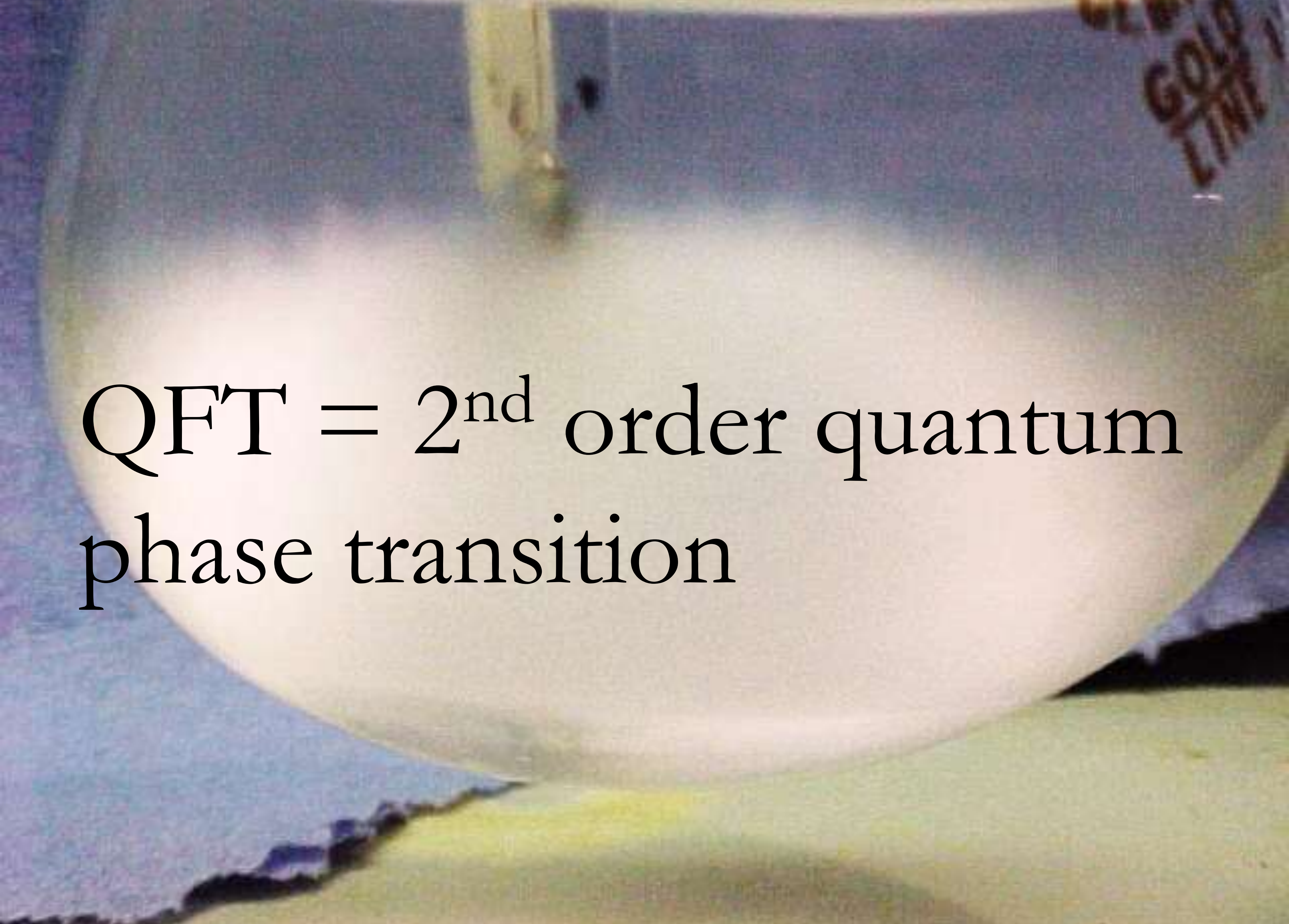
Regulated theories = lattice models



Continuous (scaling) limit:

fix **large-scale** n-pt correlation functions

Let $\epsilon \to 0$, thus $\xi \to \infty$

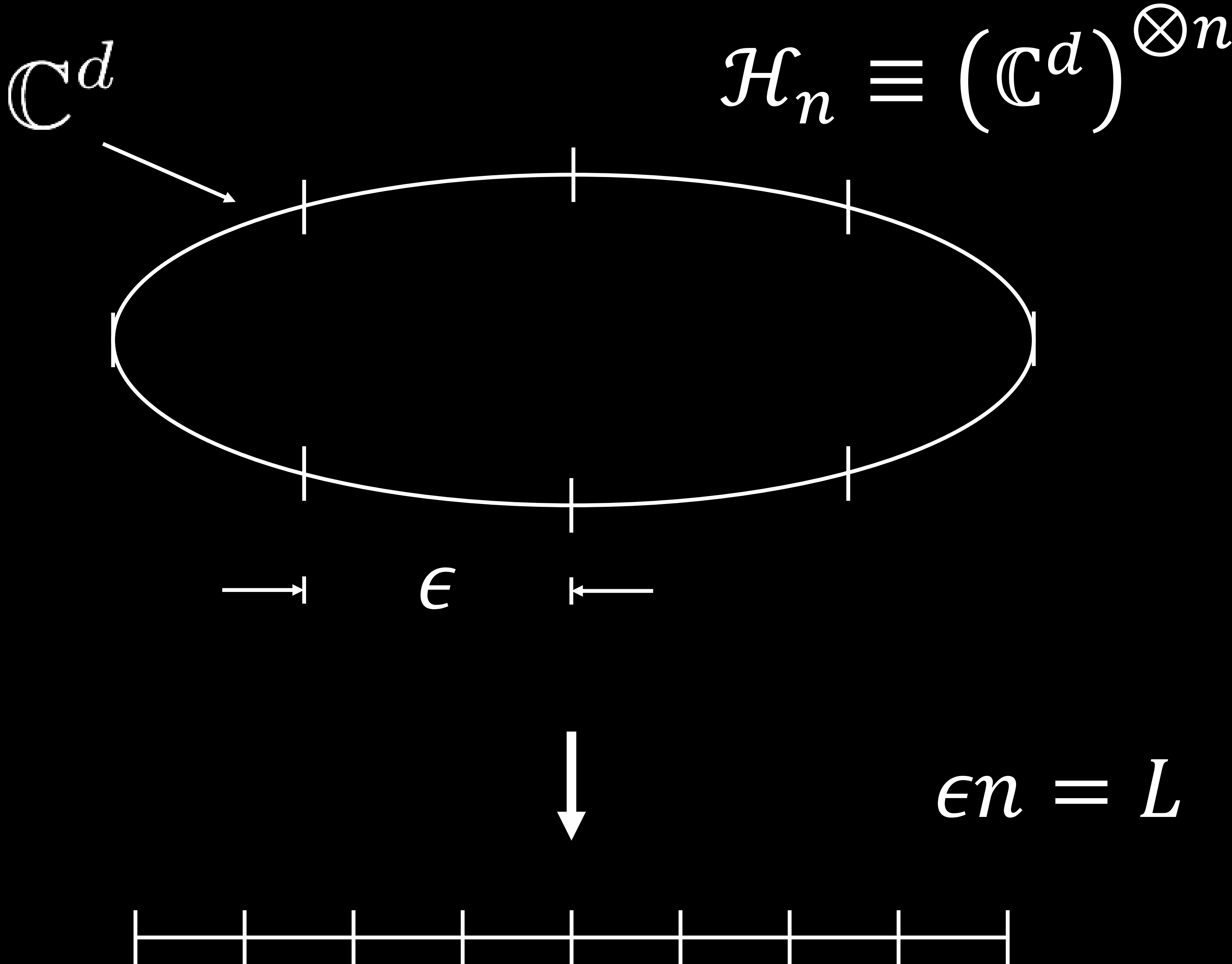

QFT = 2nd order quantum
phase transition

Effective QFT:

- Pick ϵ as small as you can
- Regulated theory is in some "convergent" sequence (which one?)
- (**Universality**)

Operator Algebraic Renormalization (OAR)

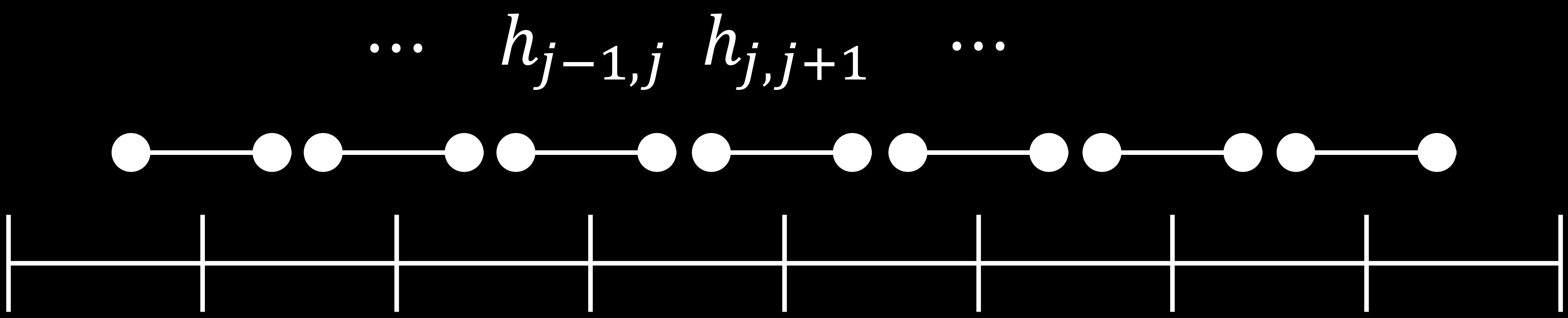
[0] M. S. Zini and Z. Wang, arXiv:1706.08497 (2017)
[1] TJO, arXiv:1901.06124 (2019)
[2] A. Brothier and A. Stottmeister, arXiv:1907.05549 (2019)
[3] A. Stottmeister, V. Morinelli, G.Morsella, and Y. Tanimoto, arXiv:2002.01442 (2020)
[4] V. Morinelli, G.Morsella, A. Stottmeister, and Y. Tanimoto, arXiv:2010.11121 (2020)
[5] TJO and A. Stottmeister, arXiv:2107.13834 (2021)
[6] TJO and A. Stottmeister, arXiv:2109.14214 (2021)



$\mathbb{C}^d$
$\mathcal{H}_n \equiv \left(\mathbb{C}^d\right)^{\bigotimes n}$
ϵ
$\epsilon n = L$

This is important!

$$H(\epsilon) = \sum_j h_{j,j+1}(\epsilon)$$

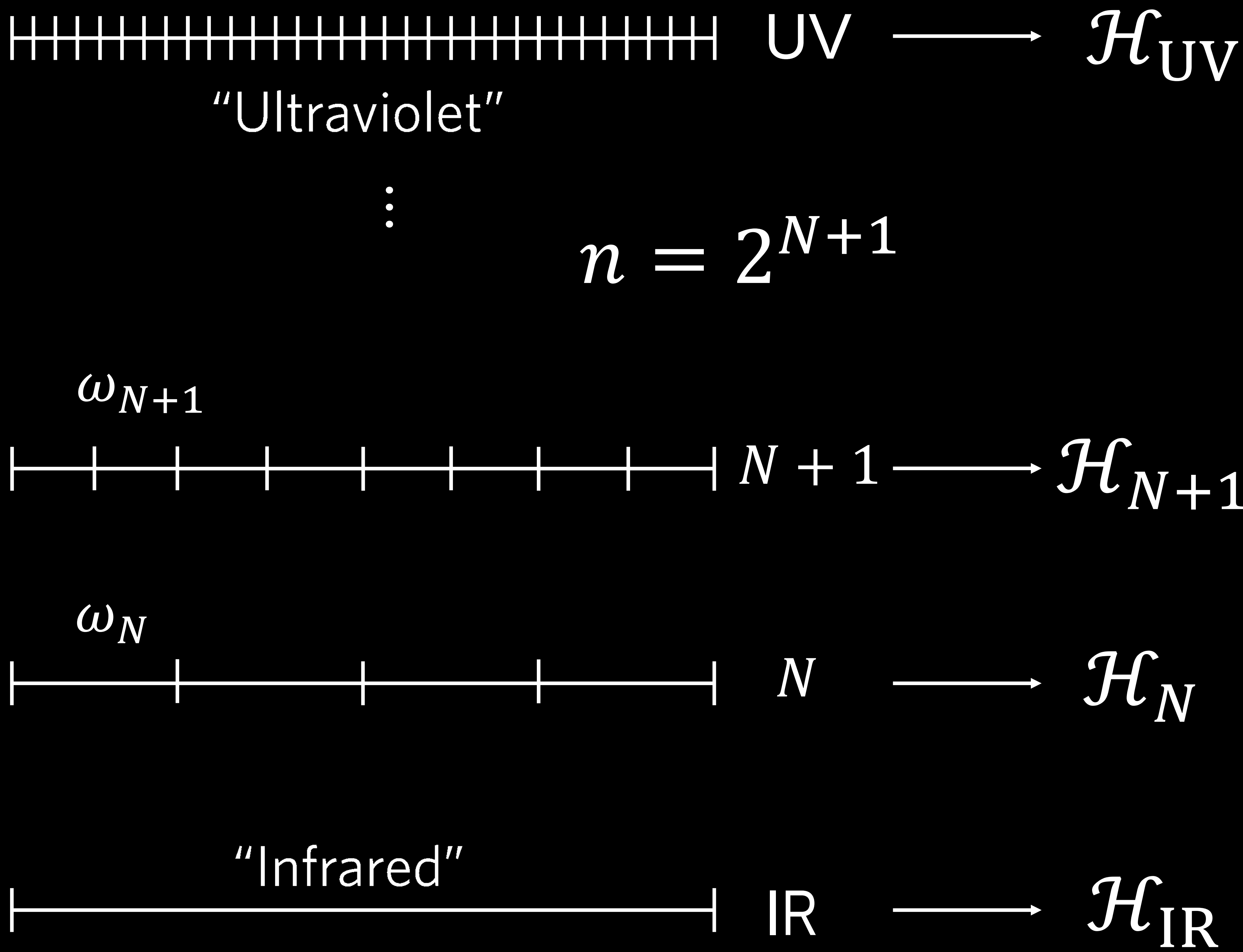


$H(\epsilon) \rightarrow$ a sequence of ground states (depending on n and ϵ):

$$\omega_\epsilon(\cdot) \equiv \langle \Omega_\epsilon | \cdot | \Omega_\epsilon \rangle$$

Choose dyadic sequence N of scales, where $n = 2^{N+1}$

$$H_0^{(N)} \equiv H(\epsilon = 2^{-N})$$



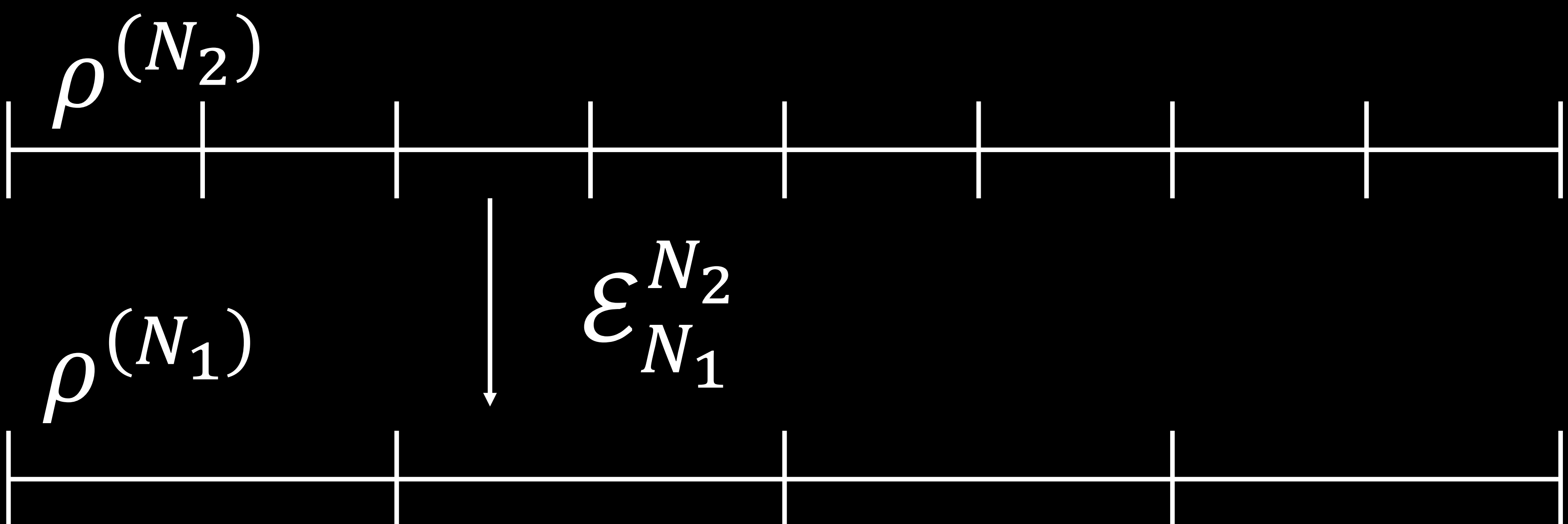
UV
$\mathcal{H}_{\mathrm{UV}}$
"Ultraviolet"
$n = 2^{N+1}$
ω_{N+1}
$N+1$
$\mathcal{H}_{N+1}$
ω_N
N
$\mathcal{H}_N$
"Infrared"
IR
$\mathcal{H}_{\mathrm{IR}}$

Continuum limit: need to compare states at different scales

(convergent sequence)

Coarse graining:

$$\mathcal{E}_{N_1}^{N_2}\left(\rho^{(N_2)}\right) = \rho^{(N_1)}$$



Renormalization Group (RG):

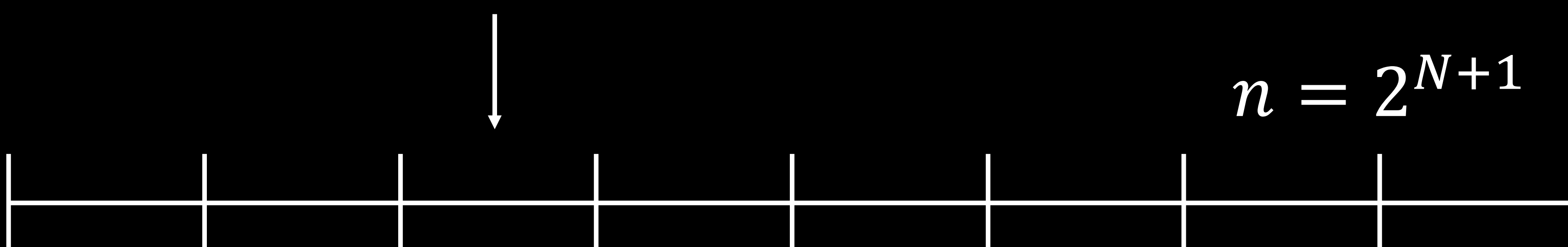
$$\left\{\mathcal{E}_{N_1}^{N_2}\right\}_{N_2>N_1}$$

Schrödinger picture

Observables (at scale N):

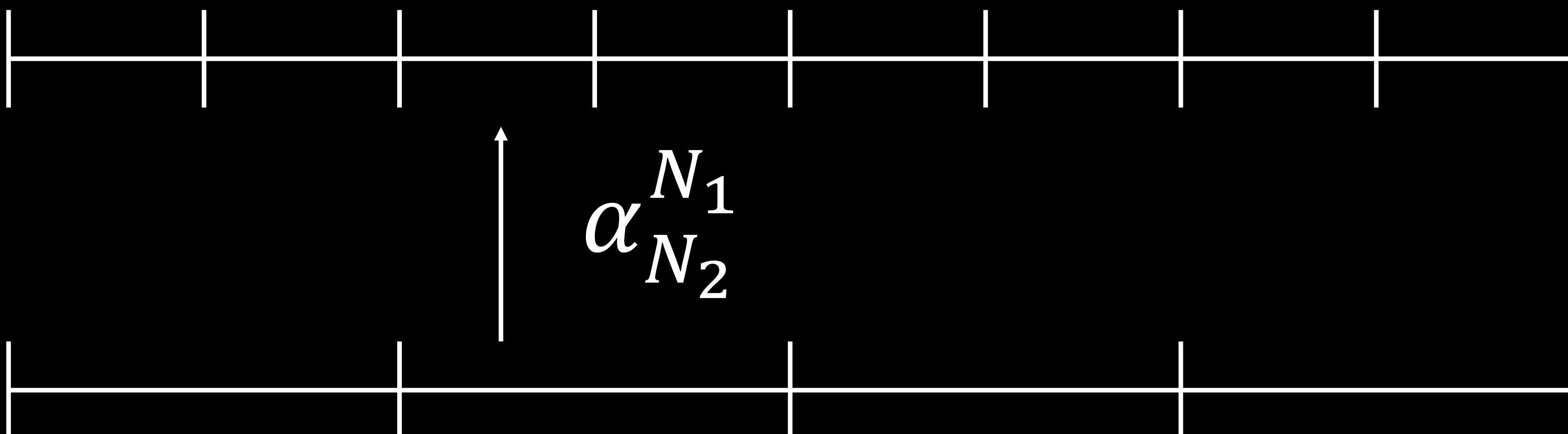
$$\mathfrak{A}_N \equiv \left(M_d(\mathbb{C})\right)^{\otimes n}$$

$M_d(\mathbb{C})$: observable algebra at site j



Heisenberg picture:

$$\alpha^{N_1}_{N_2} : \mathfrak{A}_{N_1} \to \mathfrak{A}_{N_2}$$



Renormalization Group (RG):

$$\left\{\alpha_{N_2}^{N_1}\right\}_{N_2>N_1}$$

where $\alpha_{N_3}^{N_2} \circ \alpha_{N_2}^{N_1} = \alpha_{N_3}^{N_1}$ and $\alpha_N^N = \mathcal{I}_N^N$.

Heisenberg picture

Heisenberg picture RG:

$\left\{\alpha^{N_1}_{N_2}\right\}_{N_2>N_1}$, with $\alpha^{N_2}_{N_3} \circ \alpha^{N_1}_{N_2} = \alpha^{N_1}_{N_3}$ and $\alpha^{N}_{N} = \mathcal{I}^{N}_{N}$:

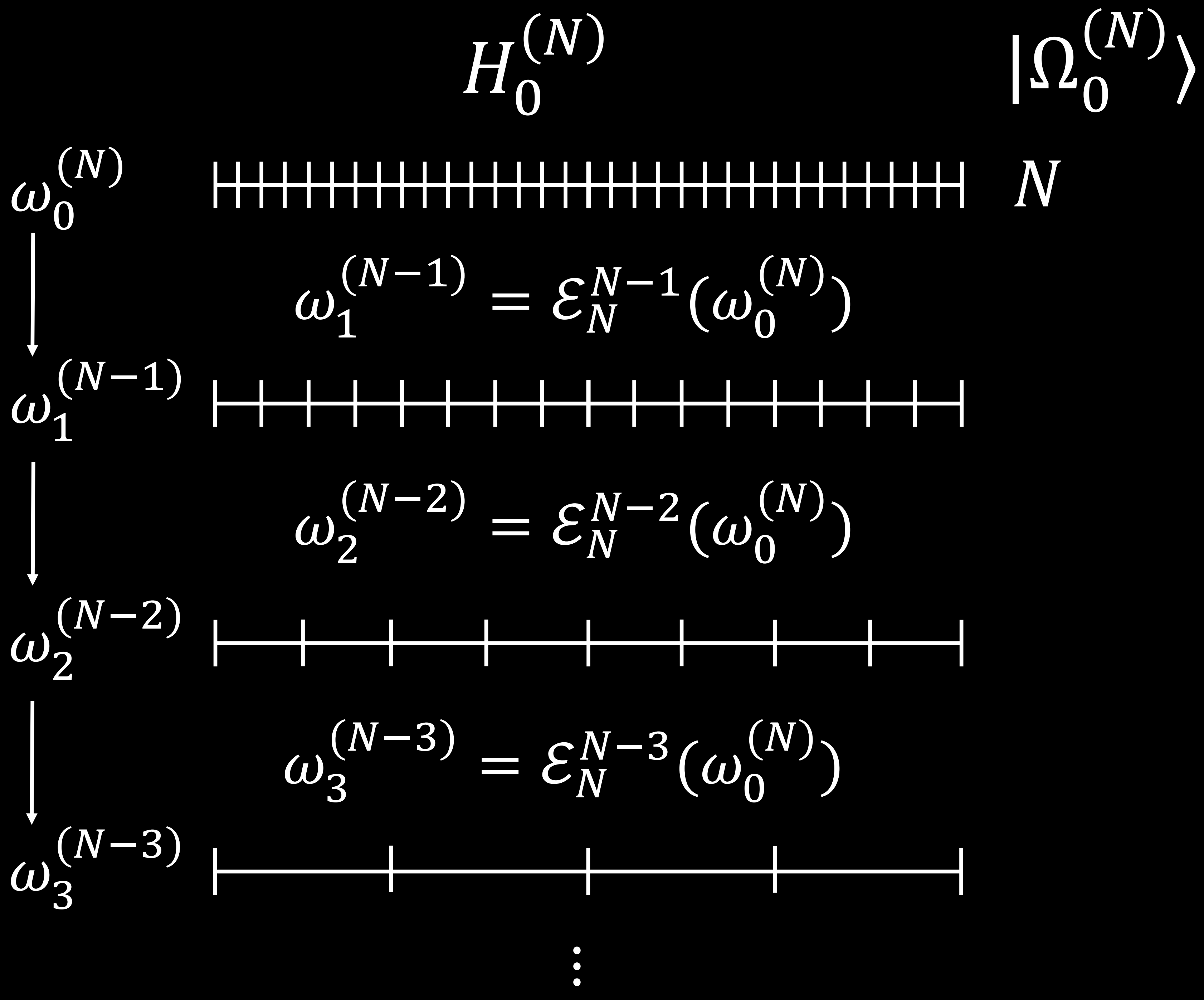
Inductive system of operator algebras $\mathfrak{A}_{\boldsymbol{N}}$

Continuum Limit Input:

$$\left\{\mathfrak{A}_N, \mathcal{H}_N, H_0^{(N)}\right\}_N$$

and

$$\left\{\alpha_{N_2}^{N_1}\right\}_{N_2 > N_1}$$


$H_0^{(N)}$
$|\Omega_0^{(N)}\rangle$
$\omega_0^{(N)}$
N
$\omega_1^{(N-1)} = \mathcal{E}_N^{N-1}(\omega_0^{(N)})$
$\omega_1^{(N-1)}$
$\omega_2^{(N-2)} = \mathcal{E}_N^{N-2}(\omega_0^{(N)})$
$\omega_2^{(N-2)}$
$\omega_3^{(N-3)} = \mathcal{E}_N^{N-3}(\omega_0^{(N)})$
$\omega_3^{(N-3)}$
$\vdots$

Wilson's triangle of renormalization:

$$
\begin{array}{ccccccccccc}
 & & & & & & \omega_0^{(N+3)} & \longrightarrow & \cdots & \omega_\infty^{(N+3)} \\
 & & & & & & \downarrow & & & \\
 & & & & \omega_0^{(N+2)} & \longrightarrow & \omega_1^{(N+2)} & \longrightarrow & \cdots & \omega_\infty^{(N+2)} \\
 & & & & \downarrow & & \downarrow & & & \\
 & & \omega_0^{(N+1)} & \longrightarrow & \omega_1^{(N+1)} & \longrightarrow & \omega_2^{(N+1)} & \longrightarrow & \cdots & \omega_\infty^{(N+1)} \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
\omega_0^{(N)} & \longrightarrow & \omega_1^{(N)} & \longrightarrow & \omega_2^{(N)} & \longrightarrow & \omega_3^{(N)} & \longrightarrow & \cdots & \omega_\infty^{(N)}
\end{array}
$$

Scaling limit state (at scale N):

$$\omega_{\infty}^{(N)} = \lim_{M \to \infty} \omega_{M}^{(N)}$$

Invariance under RG:

$$\mathcal{E}_{N_1}^{N_2}\left(\omega_{\infty}^{(N_2)}\right) = \omega_{\infty}^{(N_1)}$$

Continuum limit:

$$\omega_{\infty}^{(\infty)} = \lim_{N\to\infty} \omega_{\infty}^{(N)}$$

Trivial example:

$$H_0^{(N)} = -\sum_j \sigma_j^z$$

$$|\Omega_0^{(N)}\rangle = |0\rangle|0\rangle\cdots|0\rangle$$

Trivial example:

$$\gamma(A) \equiv \frac{1}{2}(A \otimes \mathbb{I} + \mathbb{I} \otimes A)$$

$$\alpha_{N+1}^{N} \equiv \gamma \otimes \gamma \otimes \cdots \otimes \gamma$$

Trivial example:

- We find: $|\Omega_M^{(N)}\rangle = |0\rangle|0\rangle\cdots|0\rangle$
- Therefore scaling limit is pure: $|\Omega_\infty^{(N)}\rangle = |0\rangle|0\rangle\cdots|0\rangle$

Scaling Limit (at scale N):

- **Existence is hard**
- **Depends on** $H_0^{(N)}, \left\{\alpha_{N_2}^{N_1}\right\}_{N_2>N_1}$
- Only proven (so far) for free fermions, bosons, selected spin systems

Nontrivial example:

$$H_0^{(N)} = \frac{1}{\epsilon_N} \frac{L}{2\pi} \sum_j \left(\sigma_j^z \sigma_{j+1}^z - \sigma_j^y \sigma_{j+1}^y + \lambda_N \sigma_j^x \right) + \mathrm{B.C.s}$$

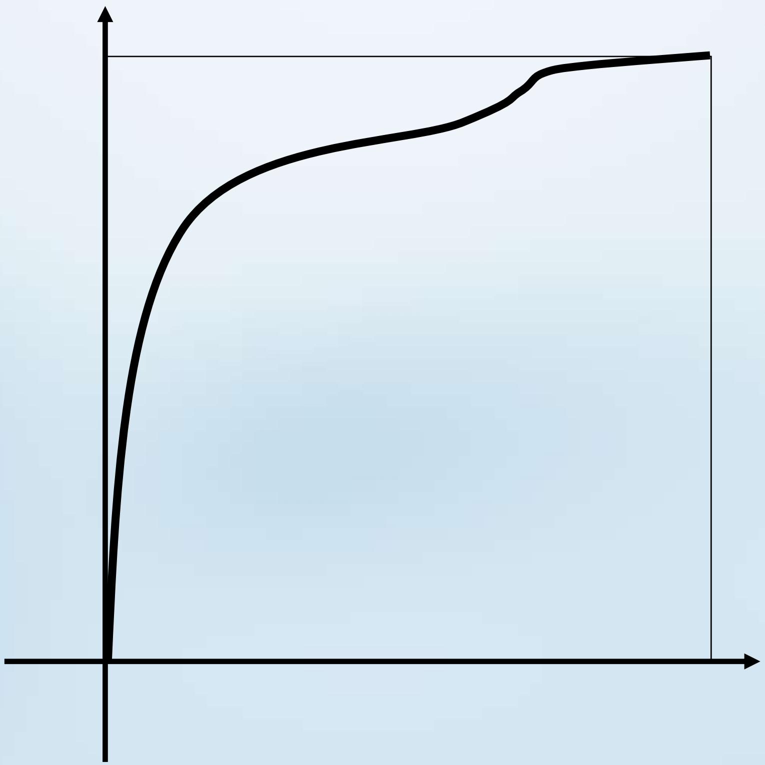
[1] T. J. Osborne and A. Stottmeister, arXiv:2109.14214 (2021)

DYNAMICS

CFT Dream: find a unitary action of $\mathrm{conf}(\mathbb{R}^{1,1})$

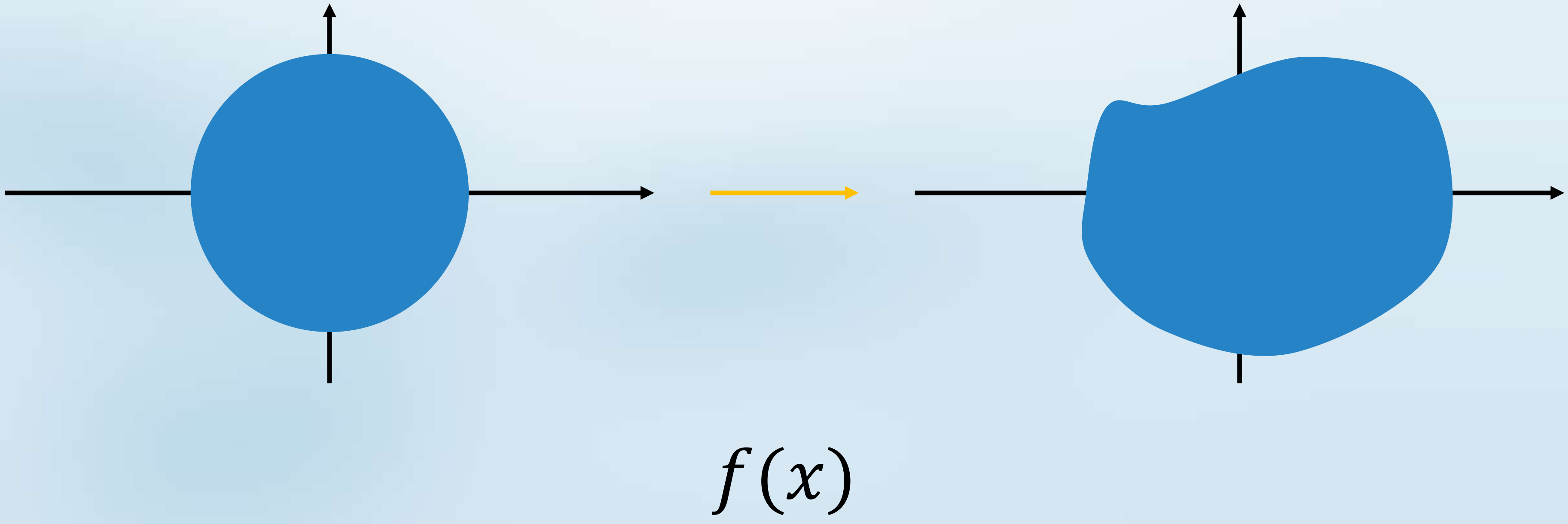
$$\mathrm{conf}(\mathbb{R}^{1,1}) \cong \mathrm{diff}_+(S^1) \times \mathrm{diff}_+(S^1)$$

$\mathrm{diff}_+(S^1)$: reparametrizations of circle under composition

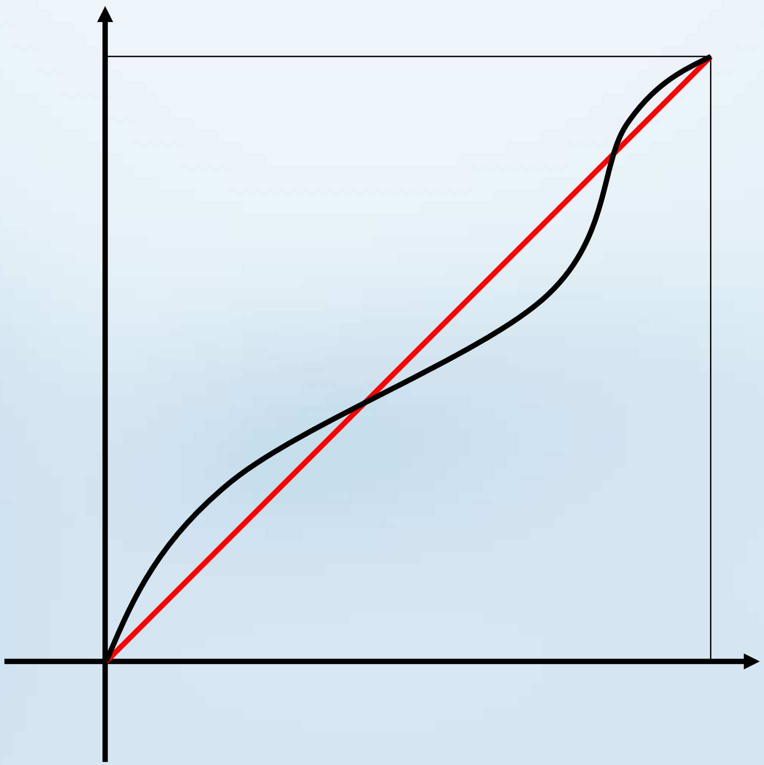


$f(x)$

$\mathrm{diff}_+(S^1)$: conformal mapping of disc

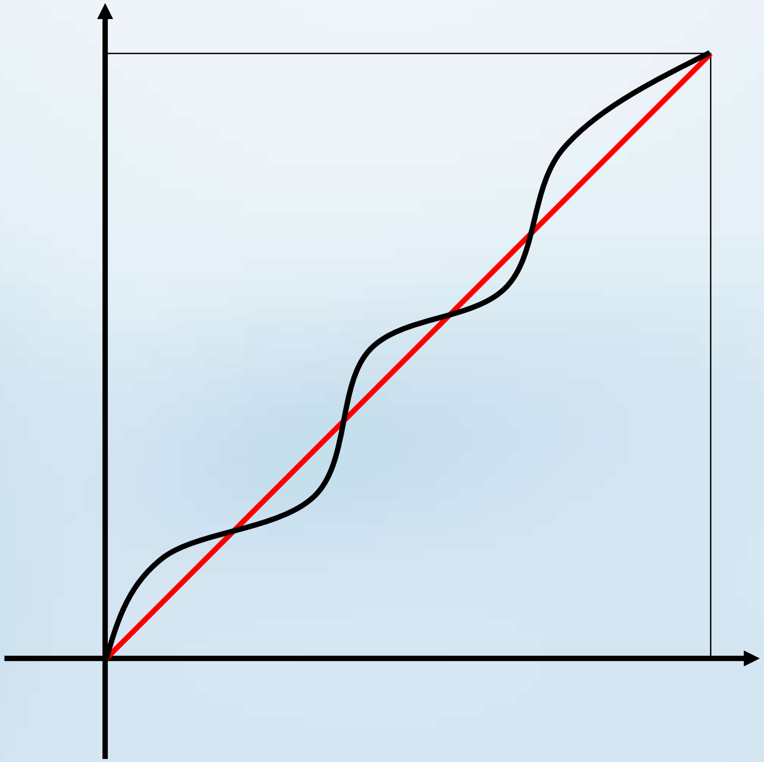


Virasoro (Witt) algebra:
infinitesimal reparametrizations



$$f(x) \approx x + \epsilon f'(x)$$

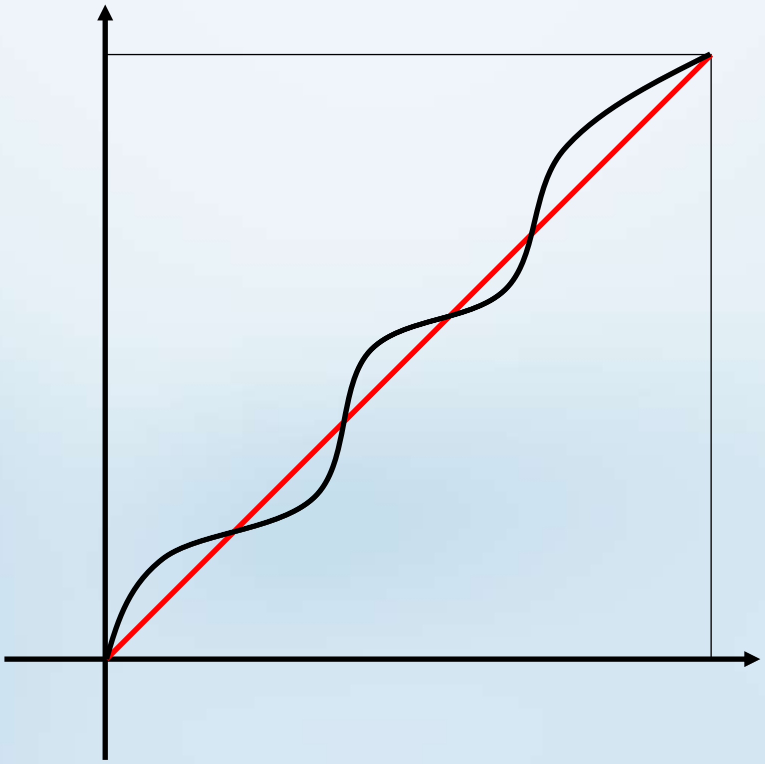
Virasoro (Witt) algebra: infinitesimal reparametrizations



$$f(x) \approx x + \epsilon\psi(x)$$

Virasoro (Witt) algebra:

infinitesimal reparametrizations



$$\psi(x) = i\,\frac{\epsilon}{2\pi}\sum_n \psi_n\, \mathrm{e}^{inx}$$

Virasoro algebra: (projective) unitary representations

$$U\colon \mathrm{diff}_+(S^1) \to \mathcal{H}$$

$$U[x + \epsilon\cos(nx)] \approx \mathbb{I} - \frac{i}{2}\epsilon(\hat{L}_n + \hat{L}_{-n})$$

Virasoro algebra: (projective) unitary representations

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

Nonuniform hamiltonians:

$$H = \int_0^{2\pi} h(x)dx$$

↓

$$H[v] = \int_0^{2\pi} v(x)h(x)dx$$

Hamiltonian density modes:

$$H = \int_0^{2\pi} h(x)dx$$

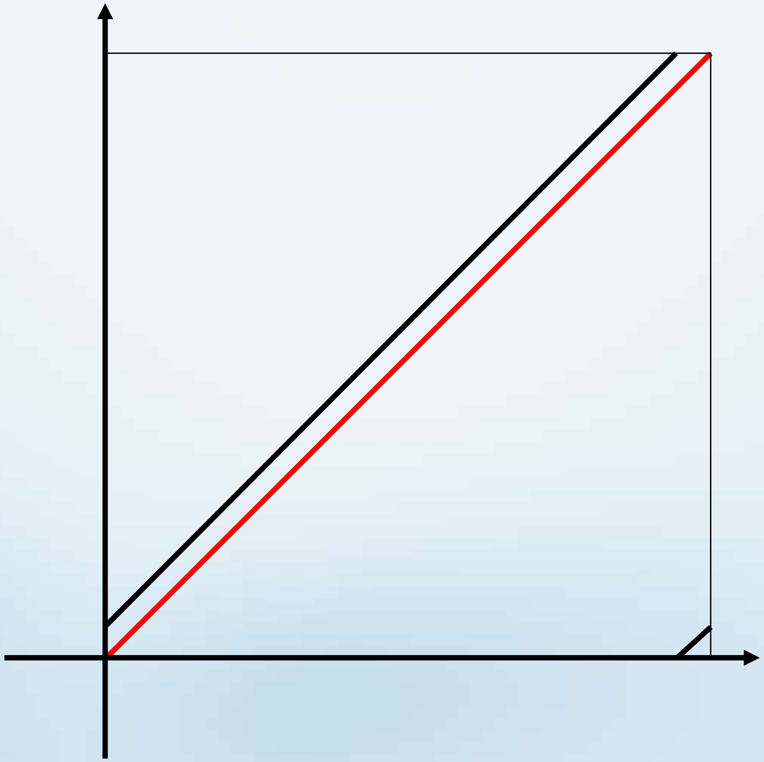
↓

$$\hat{h}(k) = \int_0^{2\pi} e^{ikx} h(x)dx$$

Hamiltonian modes = Virasoro:

$$\hat{h}(k) = \hat{L}_k + \hat{L}_{-k} - \frac{c}{12}\delta_{k,0}$$

Virasoro algebra: shifts



$$f(x) \approx x + \epsilon$$

$$U[x+\epsilon] \approx \mathbb{I} - i\epsilon \hat{L}_0$$

Problem: $\mathrm{diff}_+(S^1)$ is incompatible lattice discretisation

Koo−Saleur:

$$H_k^{(N)} = \epsilon_N \frac{L}{2\pi} \sum_j e^{ijk} h_x^{(N)}$$

$$L_k^{(N)} = \frac{1}{2}\left(H_k^{(N)} + \frac{\pi \epsilon_N}{2L \sin(\frac{1}{2}\epsilon_N\, k)} \left[H_k^{(N)}, H_0^{(N)} \right] \right) + \frac{c}{24}\delta_{k,0}$$

[1] W. M. Koo, H. Saleur, Nucl. Phys. B 426, 459 (1994).

Koo−Saleur action (scaling limit):

$$\left\{\mathfrak{A}_N, \mathcal{H}_N, H_0^{(N)}\right\}_N \qquad \left\{\alpha_{N_2}^{N_1}\right\}_{N_2>N_1}$$

$$\tau_{t;k}^{(N)}(\cdot) \equiv e^{itL_k^{(N)}}(\cdot)\, e^{-itL_k^{(N)}}$$

$$\downarrow$$

$$\tau_{t;k}^{(\infty)}(\cdot)$$

- Kinematics of QFT (OAR)
- Simulating dynamics (conf)
- Realisation on the lattice (Koo-Saleur)