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Gapped Ground State Phases of Quantum Spin Systems Examples.

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Examples

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The spin-$1/2$ Heisenberg chain

$\Gamma = \mathbb{Z}$, $n_x = 2$ for all x, nearest neighbor interaction:

$$H_{[a,b]} = -J \sum_{x=a}^{b-1} \mathbf{S}_x \cdot \mathbf{S}_{x+1}$$

$J > 0$ is the ferromagnetic chain: all translation-invariant states of the form $\omega_\phi = \bigotimes_x \langle \phi, \cdot \phi \rangle$, $\phi \in \mathbb{C}^2$, are ground states.

Goldstone Thm implies these states are gapless: $spec(H_{\omega_\phi}) = [0, \infty)$. For finite volumes $[0, L]$ gap is $O(L^{-2})$.

$J < 0$ is the antiferromagnetic chain: unique ground state in infinite volume. Lieb-Schultz-Mattis Thm implies gapless spectrum. For finite volumes $[0, L]$ gap $\leq C/L$.

Ferromagnetic XXZ model on $\Gamma = \mathbb{Z}^\nu$

$S = 1/2$, $\Delta > 1$.

$$H_\Lambda = - \sum_{\substack{x,y \in \Lambda \\ |x-y|=1}} S_x^1 S_y^1 + S_x^2 S_y^2 + \Delta S_x^3 S_y^3 .$$

This model has two translation invariant ground states and infinite families of interface ground states for all $\nu \geq 1$.

For $\nu = 1$ all these states have a positive ground state gap $= \Delta - 1$.

For $\nu > 1$, the gap above the translation invariant ground states is $\nu(\Delta - 1)$, while the spectrum above the interface ground states is gapless. Gottstein-Werner 1995, N-Koma 1996, Matsui 1997, Bolina-Contucci-N-Starr 2000, N-Spitzer-Starr 2007, ...

Generalization for spin S, $S = 1/2, 1, 3/2, ...$ have also been studied (Alcaraz-Salinas-Wreszinksi 1995, Koma-N 2001, ...).

The AKLT chain

Most famous example of isotropic gapped spin chain: the AKLT spin-1 chain (Affleck-Kennedy-Lieb-Tasaki, 1987-88).

$\Gamma = \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^3$;

$$H_{[1,L]} = \sum_{x=1}^{L-1} \left(\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right) = \sum_{x=1}^{L-1} P^{(2)}_{x,x+1}$$

$\dim\ker H_{[1,L]} = 4$ for all $L \geq 2$.

In the limit of the infinite chain, the ground state is unique, has a finite correlation length, and there is a non-vanishing gap in the spectrum above the ground state, and represents an Symmetry Protected Topological Phase (the Haldane phase).

Ground state is given by a Matrix Product State (MPS).

AKLT models

Affleck, Kennedy, Lieb, and Tasaki (1987-88) introduced a class of nearest neighbor Hamiltonians on regular lattices, later generalized by Kirillov and Korepin (1989) to general graphs G. For each $x \in G$, $\mathcal{H}_x = \mathbb{C}^{d_x}$, with $d_x =$ degree of x $+1$. The d_x- dimensional irrep of $SU(2)$ acts on $\mathcal{H}_x$. Let $z(e)$ denote the sum of the degrees of the vertices of the an edge e in G. Then

$$H_G^{\mathrm{AKLT}} = \sum_{\text{edges } e \text{ in } G} P_e^{(z(e)/2)},$$

where $P_e^{(j)}$ denoted the orthogonal projection on the states on the edge e of total spin j. Recall

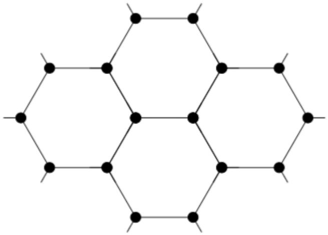
$$V_{j_1} \otimes V_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V_j.$$

AKLT model on hexagonal (honeycomb) lattice

At each vertex sits a spin of magnitude $S = 3/2$ ($\mathcal{H}_x = \mathbb{C}^4$).

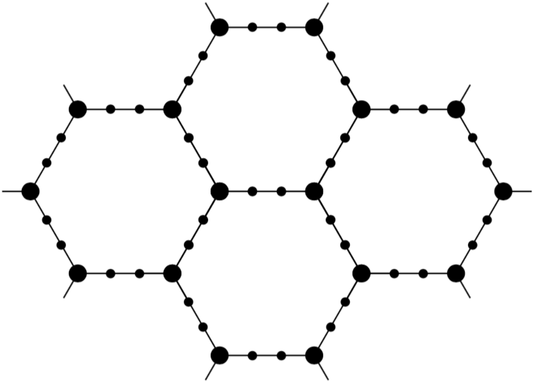
Hamiltonian:

$$H^{AKLT} = \sum_{\text{edges } \{x,y\}} h_{x,y}^{AKLT}.$$



The AKLT model on n-decorated honeycomb.

E.g.: 2-decorated hexagonal lattice:

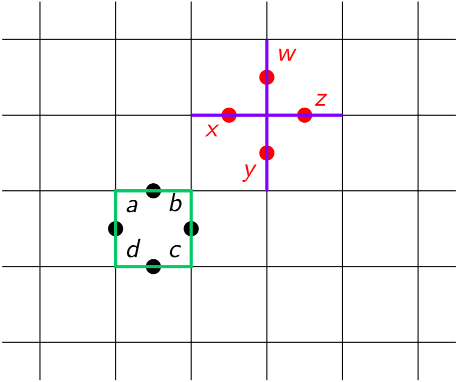


Theorem (AbdulRahman-Lemm-Lucia-N-Young, 2020)

For all $n \geq 3$, there exist $\gamma_n > 0$, such that spectral gap above the ground state of the AKLT model on an n-decorated hexagonal lattice is bounded below by γ_n.

Toric Code Hamiltonian (Kitaev 2006)

$\Gamma = \mathcal{E}(\mathbb{Z}^2)$, the edges of the square lattice; $\mathcal{A}_x = \mathbb{C}^2$, for all $x \in \Gamma$



$$H = \sum_v (\mathbb{1} - A_v)$$

$$+ \sum_f (\mathbb{1} - B_f)$$

$$A_v = \sigma^1_w \sigma^1_x \sigma^1_y \sigma^1_z$$

$$B_f = \sigma^3_a \sigma^3_b \sigma^3_c \sigma^3_d$$

On a finite finite torus $\mathbb{Z}/(L_1\mathbb{Z}) \times \mathbb{Z}/(L_1\mathbb{Z})$, the spectrum is $\{0, 4, 8, 12, \ldots\}$, and the multiplicity of the eigenvalue 0 is 4.

$O(n)$ spin chains

$O(n)$ chains: $\Gamma = \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^n$.

Recall AKLT model, $n = 3$: nearest neighbor interaction

$\Phi(\{x, x+1\}) = h_{x,x+1} = \frac{1}{2}\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \frac{1}{6}(\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2 + \frac{1}{3}\mathbb{1} = P^{(2)}_{x,x+1}$.

The general isotropic nearest neighbor interaction for $n = 3$:

$h_{x,x+1} = \cos\phi \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \sin\phi(\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2$.

Alternative way to represent the AKLT Hamiltonian in terms of 'swap' operator, T, and a rank-1 projection:

$$2P^{(2)} = T - 2P^{(0)} + \mathbb{1},$$

where $P^{(0)}$ projects onto the singlet state. There is an o.n. basis e_1, e_0, e_{-1} such that

$$\psi = \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + e_0 \otimes e_0 + e_{-1} \otimes e_{-1}).$$

This generalizes to n-dimensional spins and arbitrary coupling constants as follows

$$uT + vQ, \quad u, v \in \mathbb{R}$$

where Q is the projection to

$$\psi = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} |\alpha, \alpha\rangle.$$

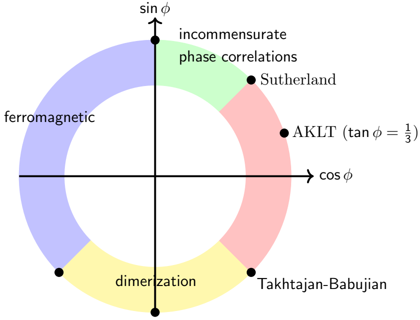


Figure: Ground state phase diagram for the $S = 1$ chain ($n = 3$) with nearest-neighbor interactions $\cos\phi \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \sin\phi (\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2$.

- $\phi = 0$ Heisenberg AF chain, Haldane phase (Haldane, 1983)
- $\tan\phi = 1/3$, AKLT point (Affleck-Kennedy-Lieb-Tasaki, 1987,1988), FF, MPS, gapped
- $\tan\phi = 1$, solvable, gapless, SU(3) invariant, (Sutherland, 1975)
- $\phi \in [\pi/2, 3\pi/2]$, ferromagnetic, FF, gapless
- $\phi = -\pi/2$, solvable, SU(3) invariant, Temperley-Lieb algebra, dimerized, gapped (Klümper; Affleck,1990)
- $\phi - = -\pi/4$ gapless, Bethe-ansatz, (Takhtajan; Babujian, 1982)

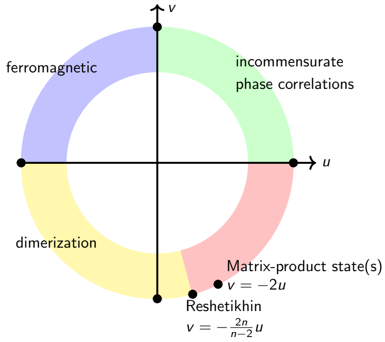
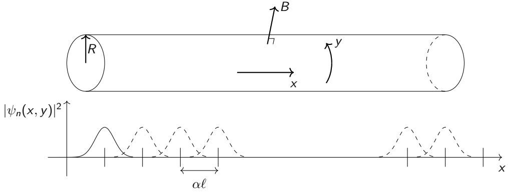


Figure: Ground state phase diagram for the chain with nearest-neighbor interactions $uT + vQ$ for $n \geq 3$, studied by Tu & Zhang, 2008.

- $v = -2nu/(n-2)$, $n \geq 3$, Bethe ansatz point (Reshetikhin, 1983)
- $v = -2u$: frustration free point, equivalent to $\perp$ projection onto symmetric vectors $\ominus$ one. Unique g.s. if n odd; two 2-periodic g.s. for even n; spectral gap in all cases and stable phase (N-Sims-Young, 2021).
- $u = 0$, $v = -1$. Equivalent to the $SU(n) - P^{(0)}$ models aka Temperley-Lieb chain; Affleck, 1990, Nepomechie-Pimenta 2016). Dimerized for all $n \geq 3$ (Aizenman, Duminil-Copin, Warzel, 2020); 'Stability' for large n (Björnberg-Mühlbacher-N-Ueltschi, 2021).

Pseudo-potential Hamiltonian for the $\nu = 1/3$ Fractional Quantum Hall Effect

Truncated Haldane model for a 1/3-filled first Landau level in a cylinder geometry:



The one-particle eigenstates ψ_n (Landau orbitals) have a Gaussian shape and are lined up along the cylinder at a spacing given by ℓ^2/R, $\ell = \sqrt{\hbar/(eB)}$, $n \in \mathbb{Z}$.

One-dimensional spin-1/2 (or spinless Fermion) Hamiltonian models the opening up of the gap in the spectrum due to interactions.

Hamilttonian with parameters $\kappa \geq 0$ and $\lambda \in \mathbb{C}$:

$$H = \sum_x \left(n_x n_{x+2} + \kappa \; q_x^* q_x \right)$$

Creation/annihilation c_x^*, c_x of Landau orbital at $x \in \mathbb{Z}$
Number operator: $n_x := c_x^* c_x$

Dipole-preserving hopping: $q_x := c_{x+1} c_{x+2} - \lambda \; c_x c_{x+3}$

Theorem (N-Young-Warzel 2020 & 2021, Young-Warzel 2022)

For all $\lambda \neq 0$ *with* $|\lambda| < 5.3548$, $\kappa \geq 0$ *there is a constant* $f\left(|\lambda|^2\right) < 1/3$ *for which*

$$\liminf_{L \to \infty} \mathrm{gap} H_{[1,L]} \geq \frac{1}{3} \min \left\{ 1, \frac{\kappa}{2 + 2\kappa|\lambda|^2}, \frac{\kappa}{1+\kappa}, \frac{\kappa}{2(1+2|\lambda|^2)} \left(1 - \sqrt{3 f(|\lambda|^2)} \right)^2 \right\} > 0 \,.$$

Note that the physical range is $|\lambda| \in [0, 3]$.

Product Vacua with Boundary States (PVBS)

A model with a gap when defined on $\mathbb{Z}^\nu, \nu \geq 2$, but gapless spectrum on certain half-spaces:

At each site $n_x = 2$; o.n.b. $\{|0\rangle, |1\rangle\}$, Let $e_1, \ldots, e_\nu$ be the canonical basis vectors of $\mathbb{Z}^\nu \subset \mathbb{R}^\nu$. The interaction is nearest neighbor: $h_{x,x+e_j}$, with $j = 1, \ldots, \nu$, such that $x, x + e_j \in \Lambda$. depending on parameters $\lambda_j \in (0, \infty)$, $j = 1, \ldots, \nu$, and are defined by

$$h_{x,x+e_j} = |\phi^{(\lambda_j)}\rangle\langle\phi^{(\lambda_j)}| + |11\rangle\langle 11|, \tag{1}$$

where $\phi^{(\lambda)} = (|01\rangle - \lambda|10\rangle)/\sqrt{1+\lambda^2}$, for $\lambda \in (0, \infty)$. The Hamiltonian is then

$$H_\Lambda = \sum_{j=1}^{\nu} \sum_{\substack{x\in\Lambda \\ \text{s.t. } x+e_j\in\Lambda}} h_{x,x+e_j}, \tag{2}$$

which is frustration-free and translation invariant.
Let γ_D be the ground state gap of the GNS Hamiltonian, H^D, in the unique ground state of this model defined on infinite half spaces bounded by a hyperplane containing the origin, that is subsets $D \subset \mathbb{Z}^\nu$ determined by a unit vector $m \in \mathbb{R}^\nu$ (the inward normal) as follows:
$D := \{x \in \mathbb{Z}^\nu : m \cdot x \geq 0\}$.

If $\nu = 1$, the model is gapless if $\lambda = 1$ and gapped otherwise (Bachmann-N, 2012).

For $\nu \geq 2$, the positivity of $\gamma(D)$ is determined by the angle, θ, between the vectors m and $-\log \boldsymbol{\lambda}$.
Define $c(v) := \min\{|v_j| : v_j \neq 0\}, v \in \mathbb{R}^\nu$.

Theorem (Bachmann-Hamza-N-Young 2015, Bishop-N-Young 2016)

(i) For all $\nu \geq 2$, $\lambda_1, \ldots, \lambda_\nu \in (0, \infty)$, and unit vectors $m \in \mathbb{R}^\nu$ such that $m \cdot \log \boldsymbol{\lambda} < 0$, one has the following upper bound:

$$\gamma(D) \leq \frac{2(d-1)}{c(m)c(\boldsymbol{\lambda})^2} \| \log \boldsymbol{\lambda} \| |\sin(\theta)|, \tag{3}$$

where θ is the angle between the vectors $-m$ and $\log \boldsymbol{\lambda}$. In particular, the gap vanishes if $\theta = 0$.
(ii) If $\log \boldsymbol{\lambda} \neq -\| \log \boldsymbol{\lambda} \| m$, then $\gamma(D) > 0$.