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EXPLORING 2D SYSTEMS WITH TENSOR NETWORKS (TTN & PEPS)

Göttingen – 20.11.2025

- ★ Physics in 2D is very interesting & challenging
 - ➔ Hubbard model for High-Tc superconductors still puzzling
 - ➔ quantum Hall effect(s) & topological matter
 - ➔ frustrated systems & spin liquids (fractionalization)
 - ➔ rich phenomenology out-of-equilibrium (interfaces, ...)
 - ➔ lack of analytics & numerics wrt 1D (& 3D... mean-field-*ish*)

A MENU OF TENSOR NETWORKS FOR 2D

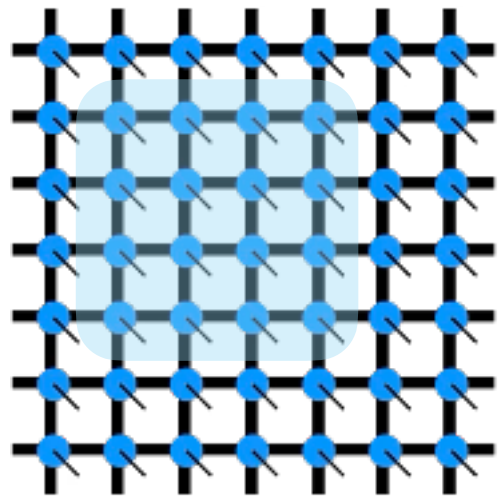
3

★ Physics in 2D is very interesting & challenging

? What tensor networks should we use to study it?

Good theory tells:

PEPS



Verstraete & Cirac, cond-mat/0407066v1

- ✓ entang. boundary law
- no canonical form
- very costly contraction
- tricky approx. optimization

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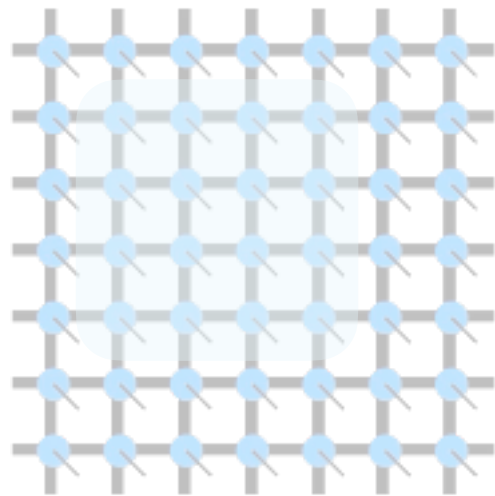
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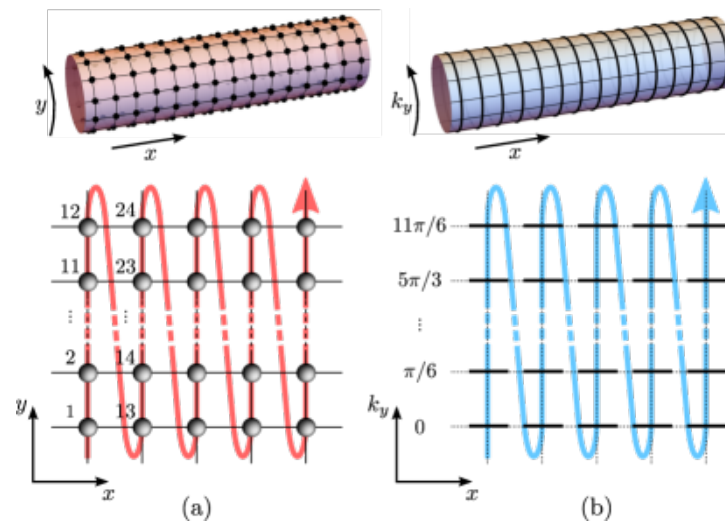
Verstraete & Cirac, cond-mat/0407066v1

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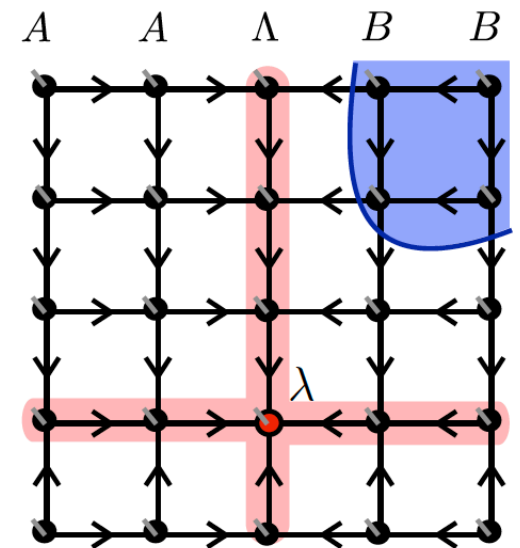
Practise often turns out to be:

cylinder-MPS



Motruk et al, PRB 93, 155139 ('16)

isoTNS



Zaletel & Pollmann, PRL 124, 037201 ('20)

A MENU OF TENSOR NETWORKS FOR 2D

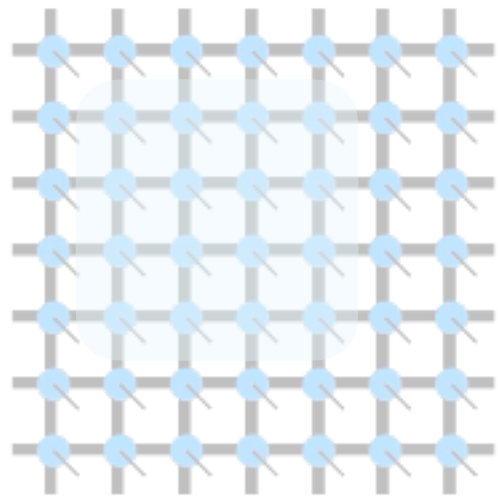
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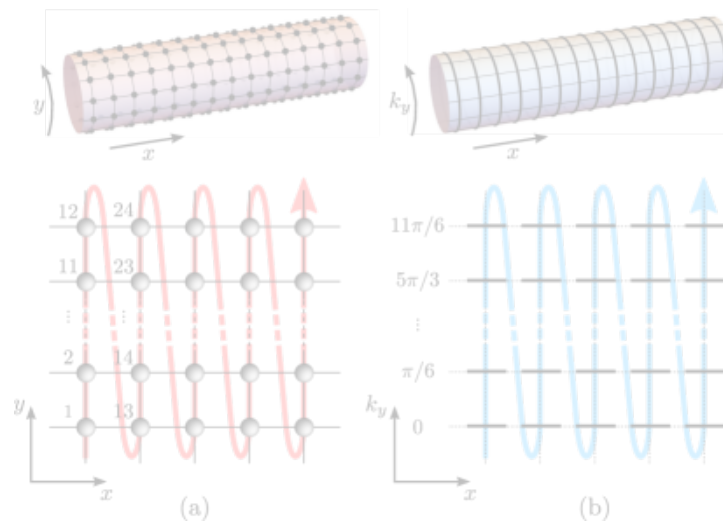
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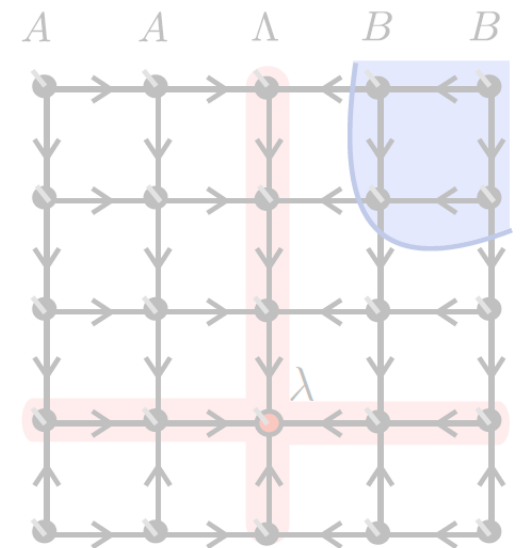
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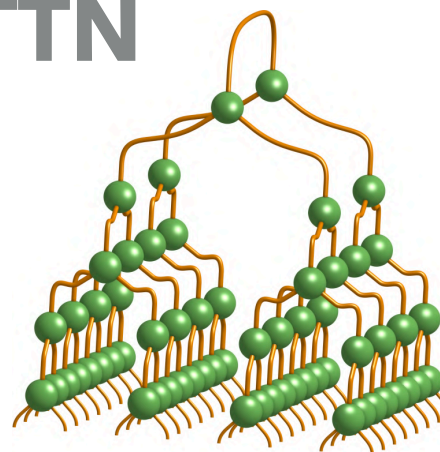
Motruk et al, PRB 93, 155139 ('16)

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TTN



- ~ area law only on average
- ✓ canonical form
- ✓ cheap contraction $O(\chi^4)$
- ✓ MPS-like optimization
- ✓ PBC at no extra cost

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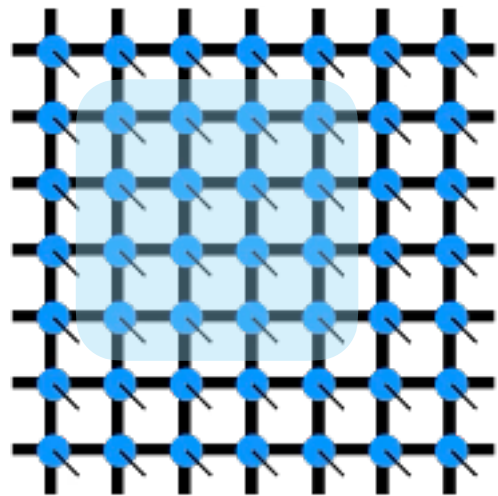
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- ? What tensor networks should we use to study it?

Disclaimer:
no NQS here :-)

Good theory tells:

PEPS

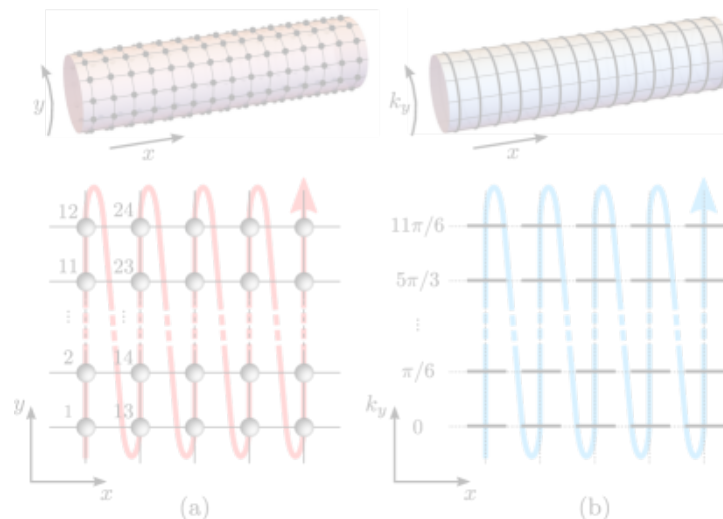


Verstraete & Cirac, cond-mat/0407066v1

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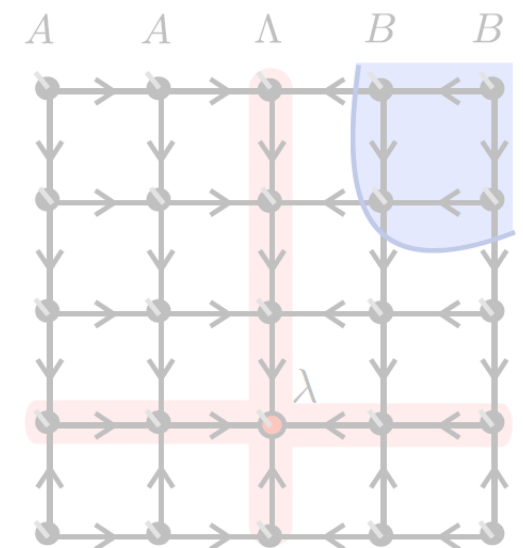
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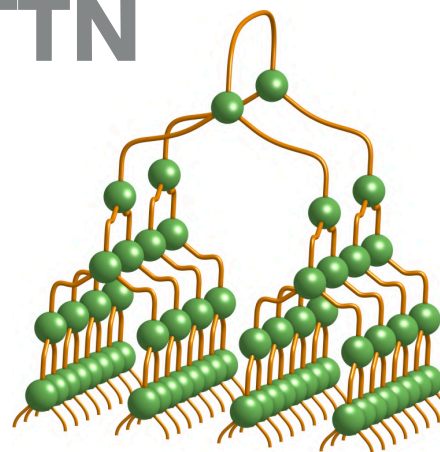
Motruk et al, PRB 93, 155139 ('16)

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Zaletel & Pollmann,
PRL 124, 037201 ('20)

TTN



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- * Overview of TN approaches for 2D
- * Tree Tensor Networks (TTN):
 - * Basic definitions & operations
 - * App 1: statics of (bosonic) fractional quantum Hall
 - * App 2: dynamics of interfaces (2D Ising)
 - * other applications & extensions
- * Projected Entangled Pair States (PEPS):
 - * Variational algorithm for infinite systems
 - * App: statics of (bosonic) fractional quantum Hall
 - * Stochastic reconfiguration for finite systems
- * Conclusion

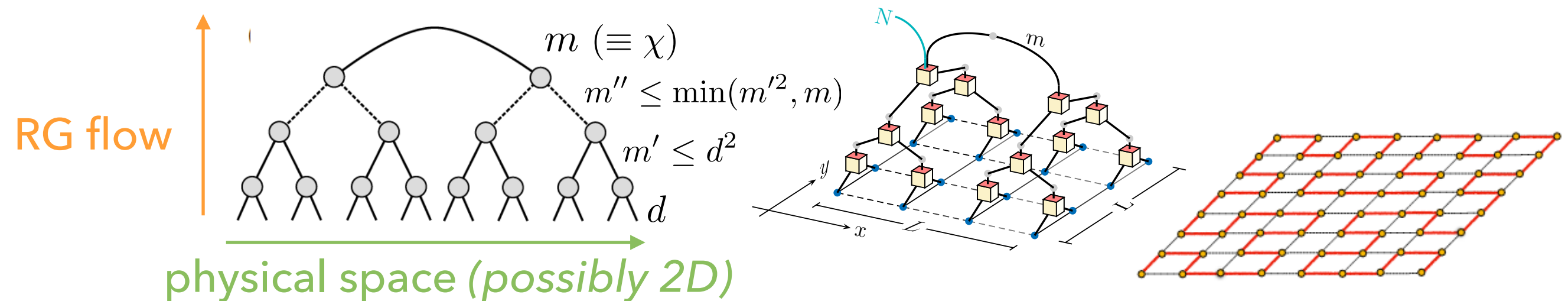
TREE TENSOR NETWORKS (TTN)

BASIC DEFINITIONS & OPERATIONS

P. Silvi, et al. ([MR](#)) SciPost Phys. Lect. Notes 8 (2019)

BASIC IDEAS OF TTN

6



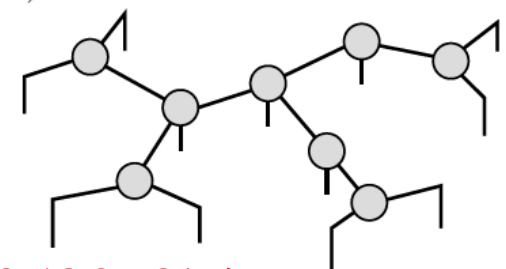
Shi et al., PRA 74, 022320 ('06) / Tagliacozzo et al., PRB 80, 235127 ('09) / Ferris, PRB 87, 125139 ('13) / ...
+ a lot by Montangero's group in recent years (e.g., T. Feller's PhD thesis)

- ✓ loop-free structures (just like MPS)
- ✓ better coverage of the space / approx. boundary law
- ✓ allow for power-law correlations (at least on average)
- ✓ tall-skinny tensors can fruitfully exploit GPU-features

Similar to multi-configuration time-dep. Hartree approach in q-chemistry

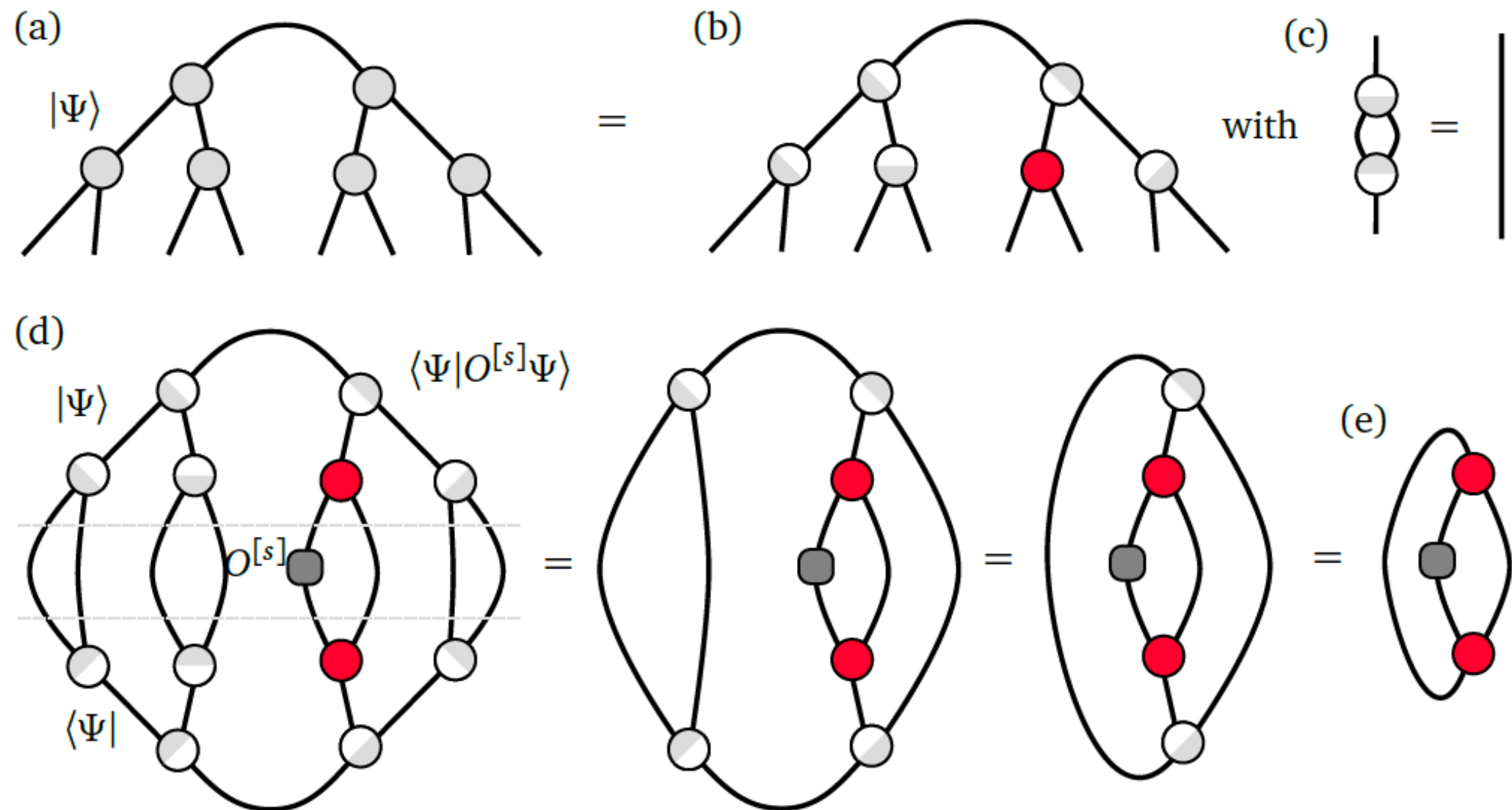
Morrison & Fischer PRA 35, 2429 ('87) / Lode et al., RMP 92, 011001 ('20)

! Different from TN for tree-like structures (e.g., molecules)



Murg, et al., PRB 82, 205105 (2010) / Nakatani & Chan, J. Chem. Phys. 138, 134113 (2013) / ...

A loop-free TTN shares all good properties with MPS:
e.g., possibility of an isometric gauge with an orthogonality center



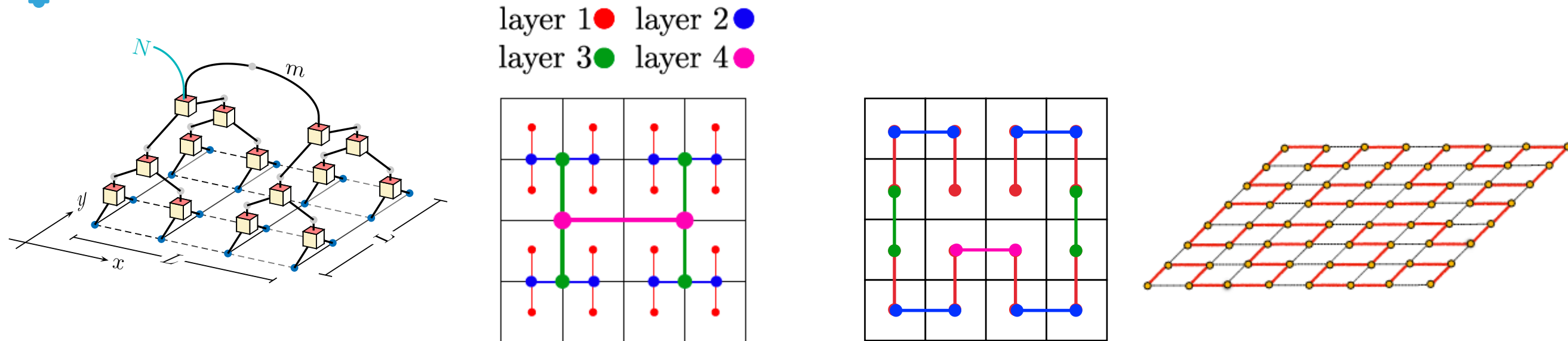
and also the possibility of a canonical gauge (*not used in practice*)

TTN FEATURES

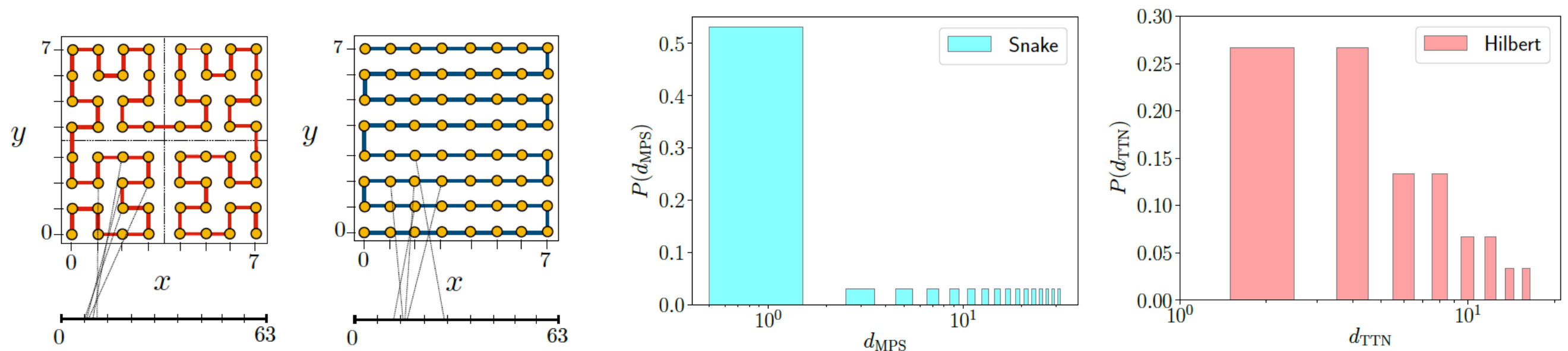
8



apply RG alternatively along x & y $\rightarrow$ 1D mapping along Hilbert curve



✓ much shorter average graph distance for physically close-by sites



worse case scales $\simeq \log L$ instead of $\simeq L$

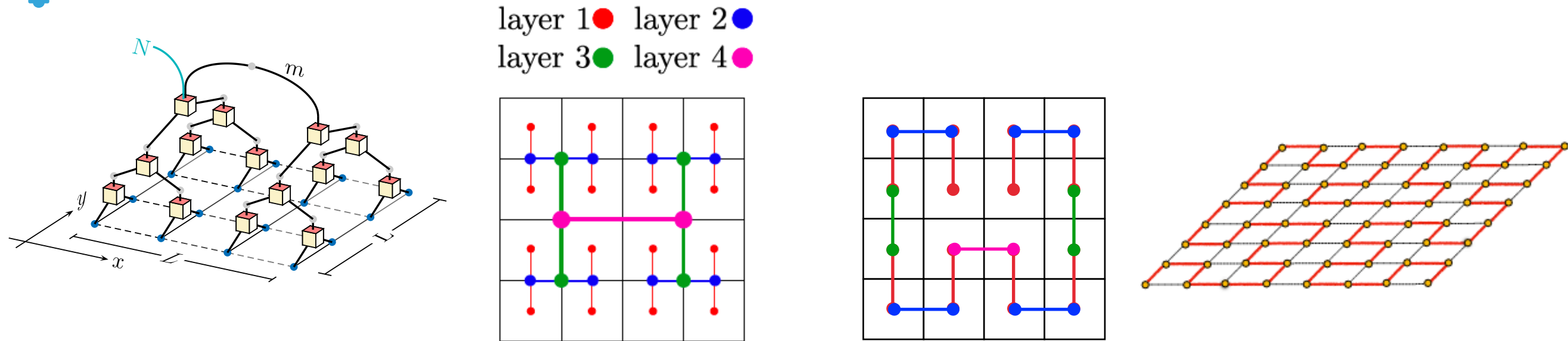
Cataldi et al., Quantum 5, 556 (2021)

TTN FEATURES

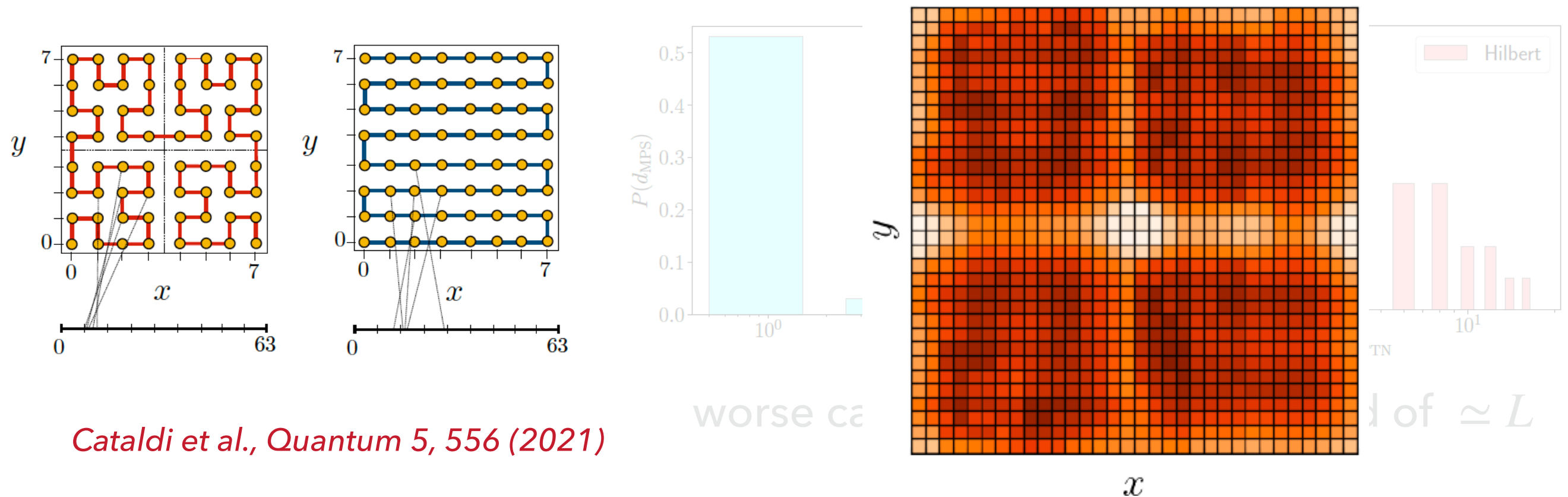
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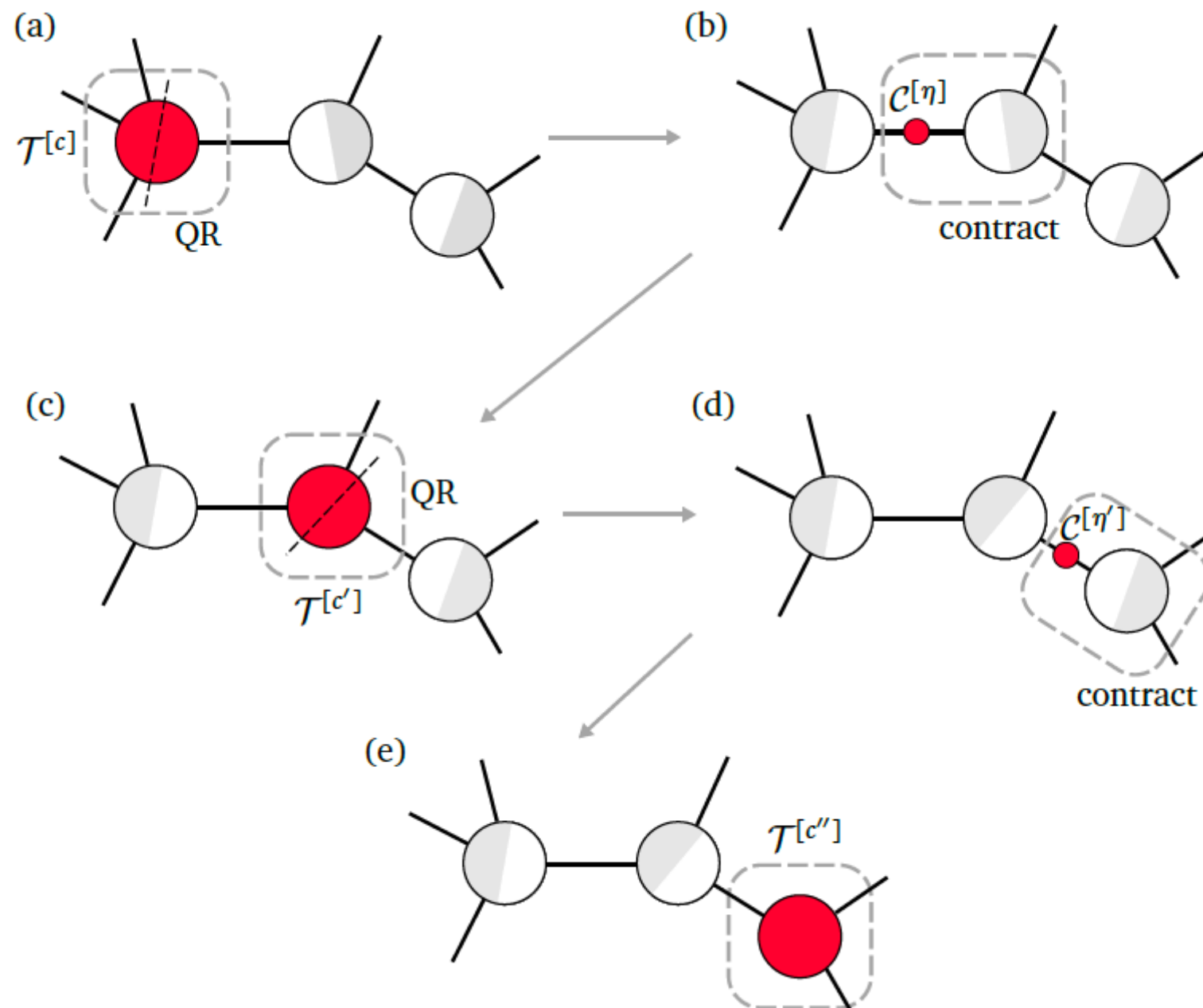


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Cataldi et al., Quantum 5, 556 (2021)

Move the **center** of the **isometric gauge** via sequential QR



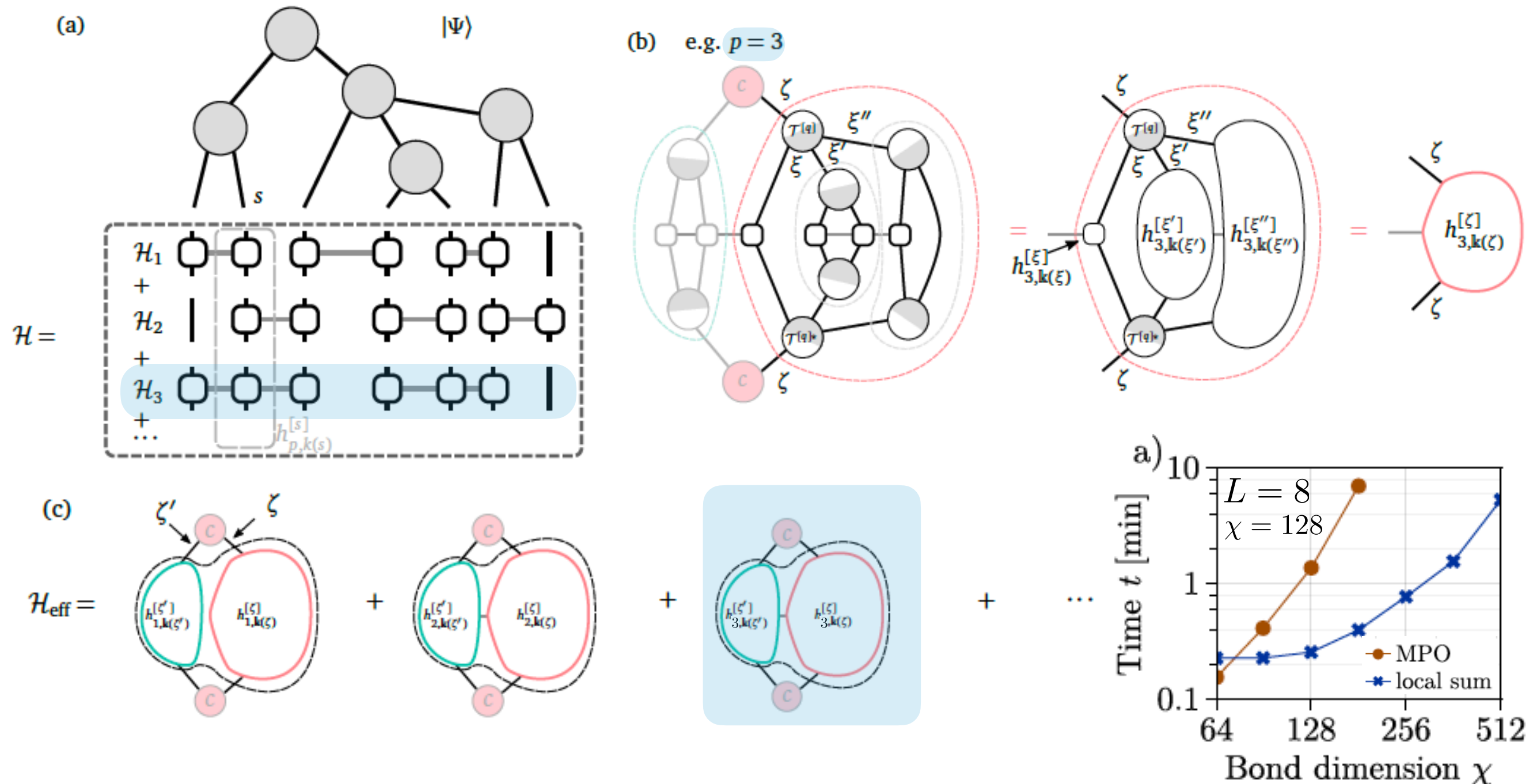
TTN BASIC OPS: EFFECTIVE HAMILTONIAN

10

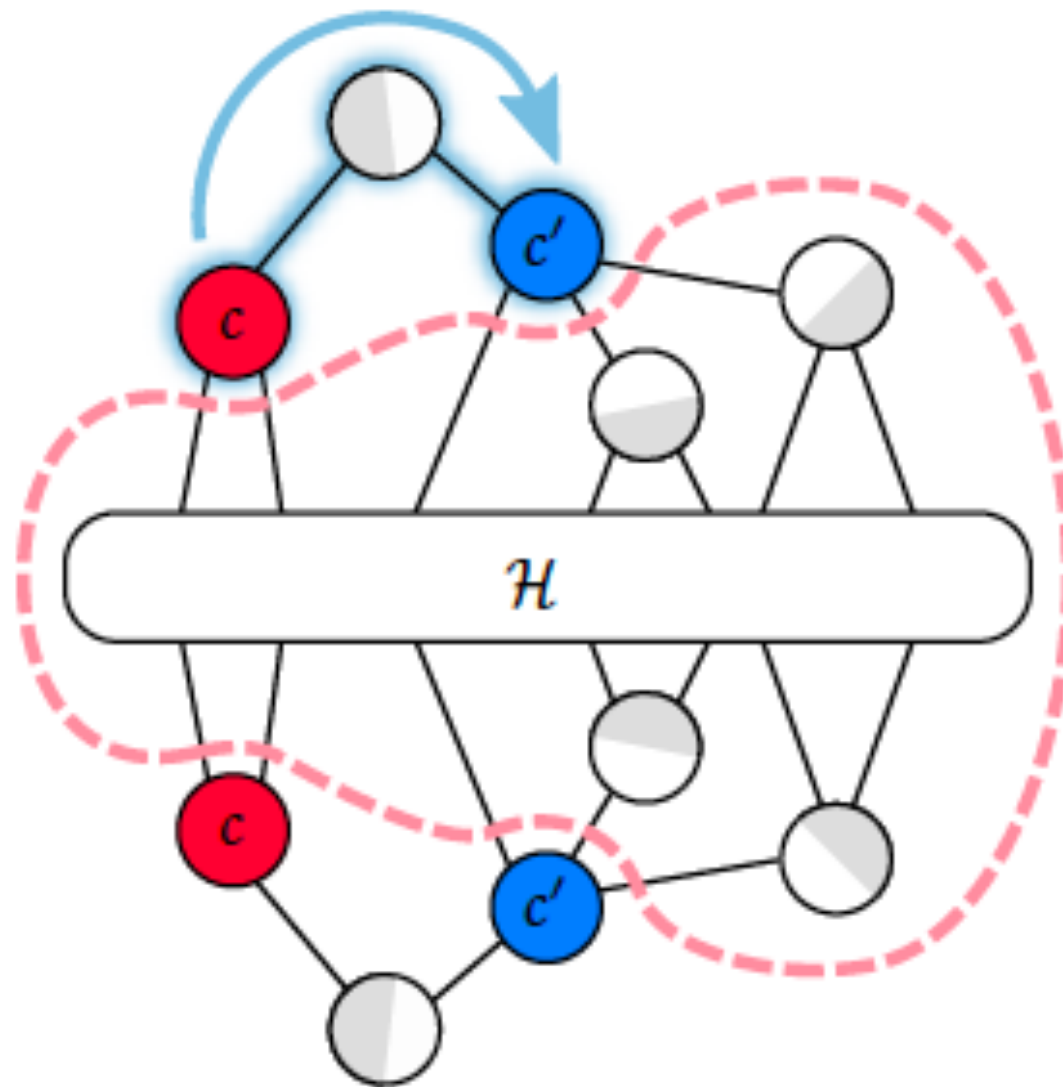
P. Silvi, et al. ([MR](#)) SciPost Phys. Lect. Notes 8 (2019)

Use Tensor Product Operators & book-keeping

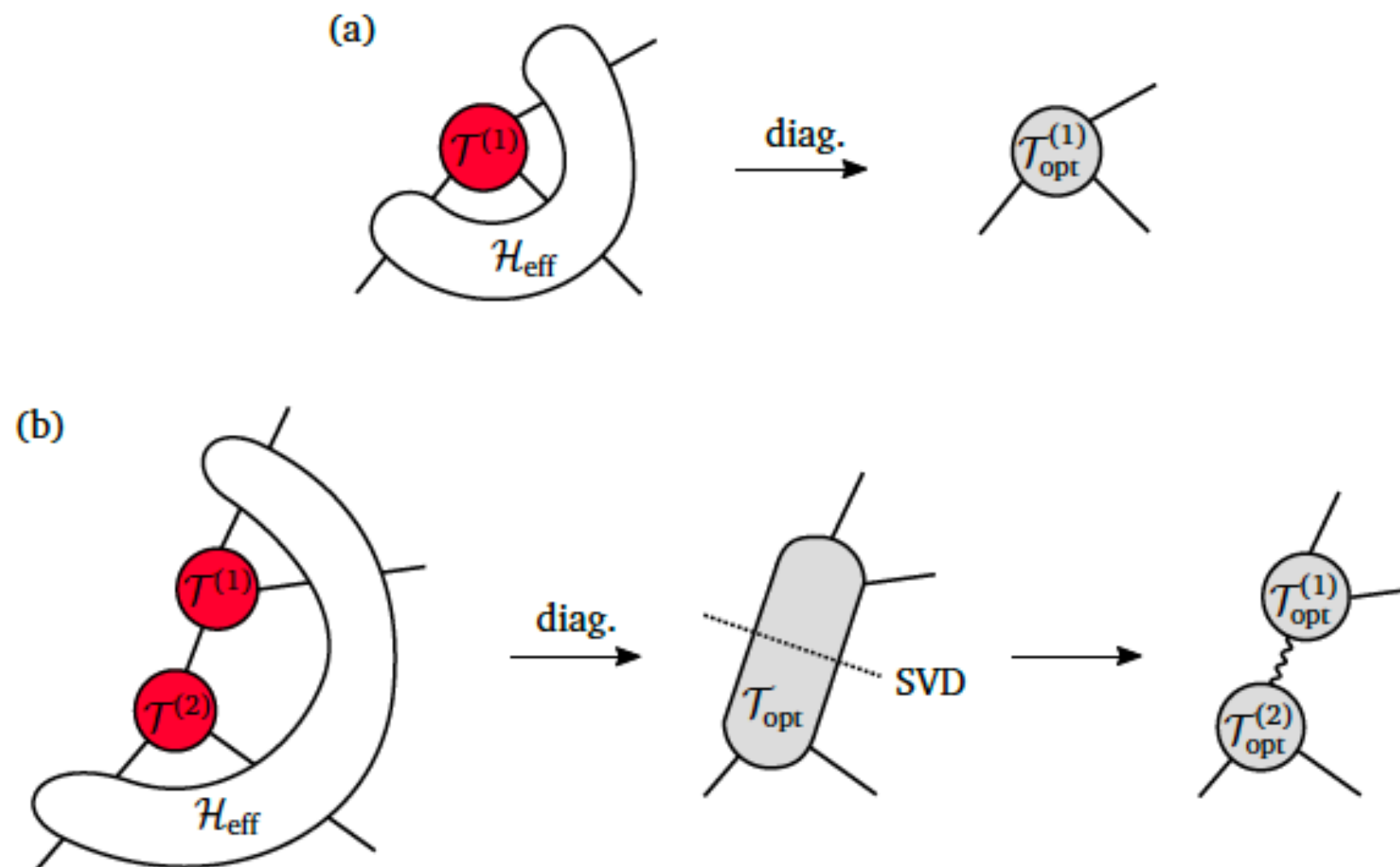
(MPO goes also transparently through, but bond-dim. grows with L)



Update effective Hamiltonian after center move
(only requires few new contractions, like MPS)



Optimize via *standard* eigenvalue problems
& **compress** via *singular value decomposition* (SVD)

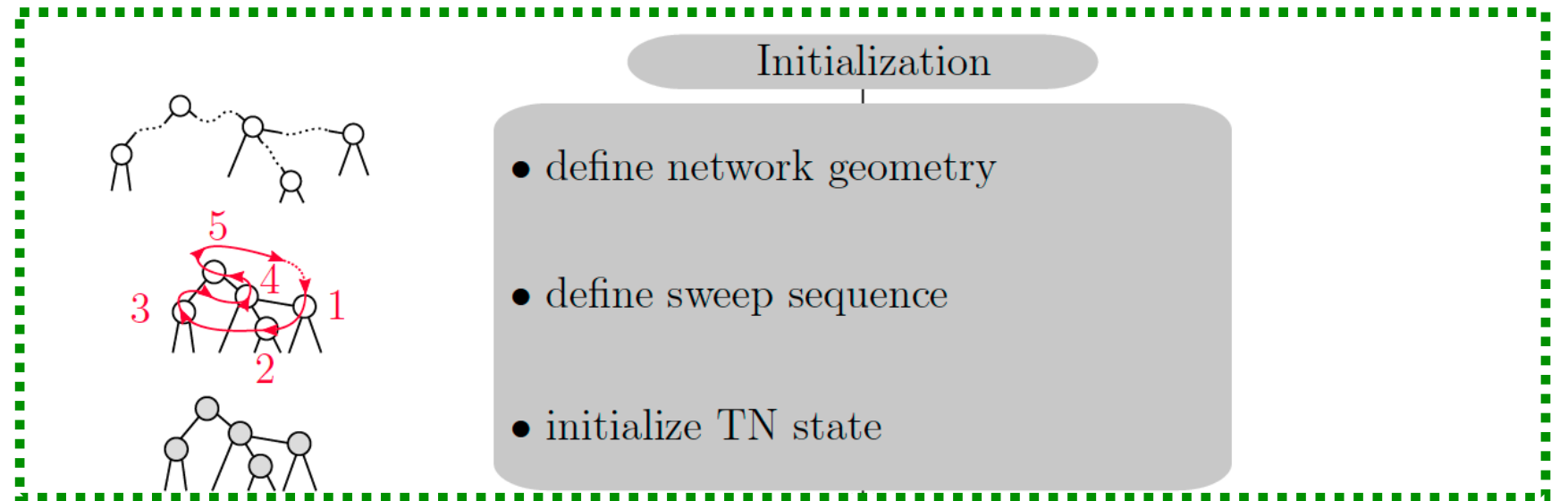
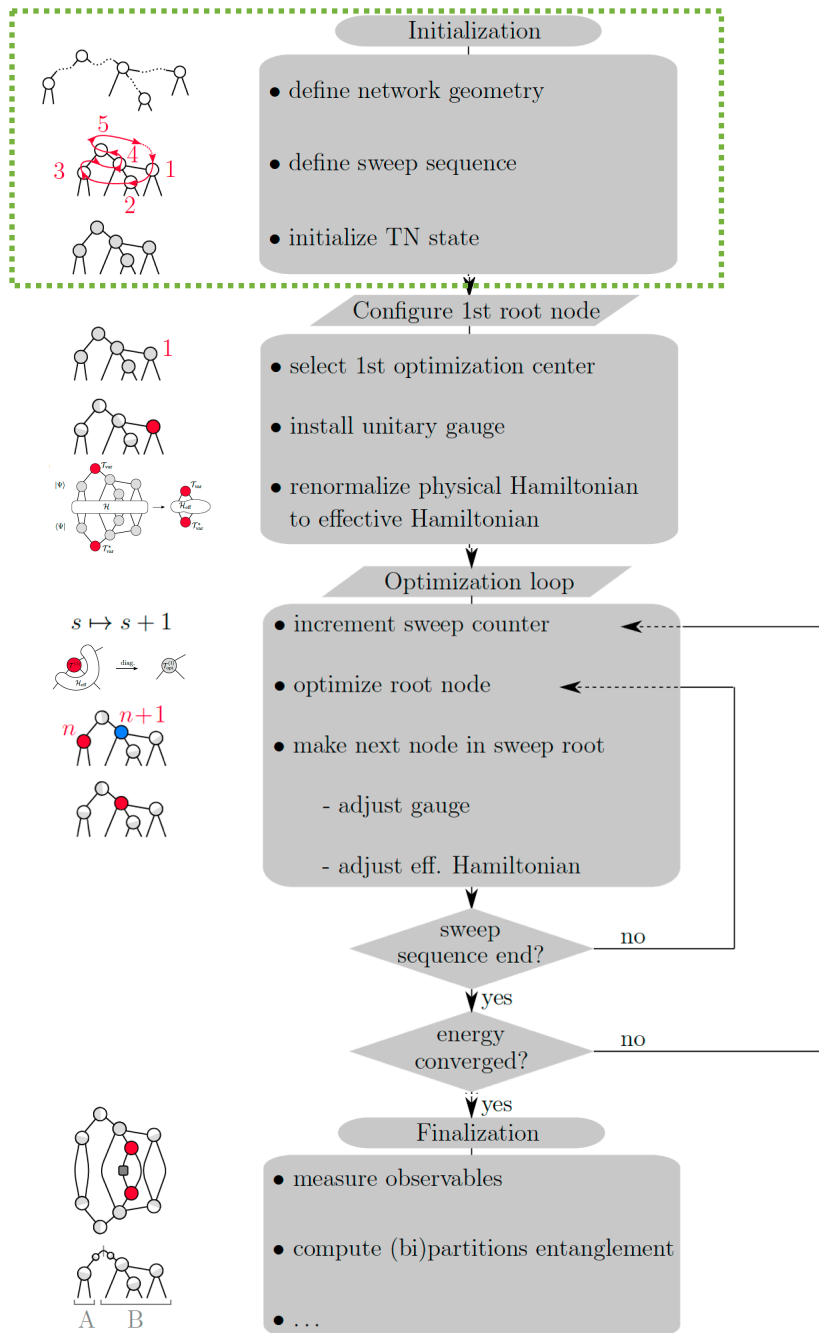


Of course, possibility of defining an hybrid strategy (expander) to cope with symmetry sectors at a cheaper cost (as in MPS)...

TTN GROUND STATE SEARCH

13

P. Silvi, et al. (MR) SciPost Phys. Lect. Notes 8 (2019)

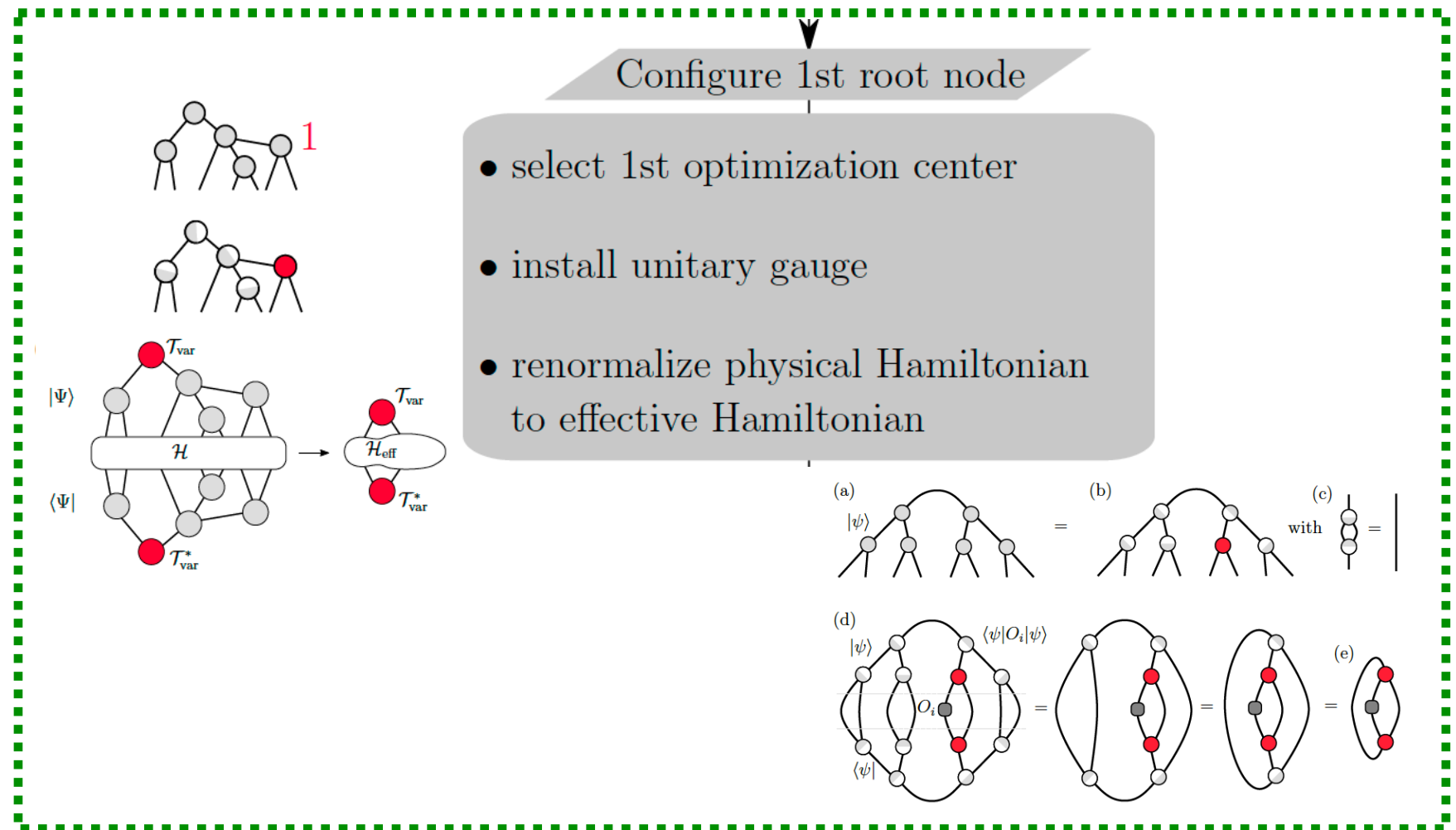
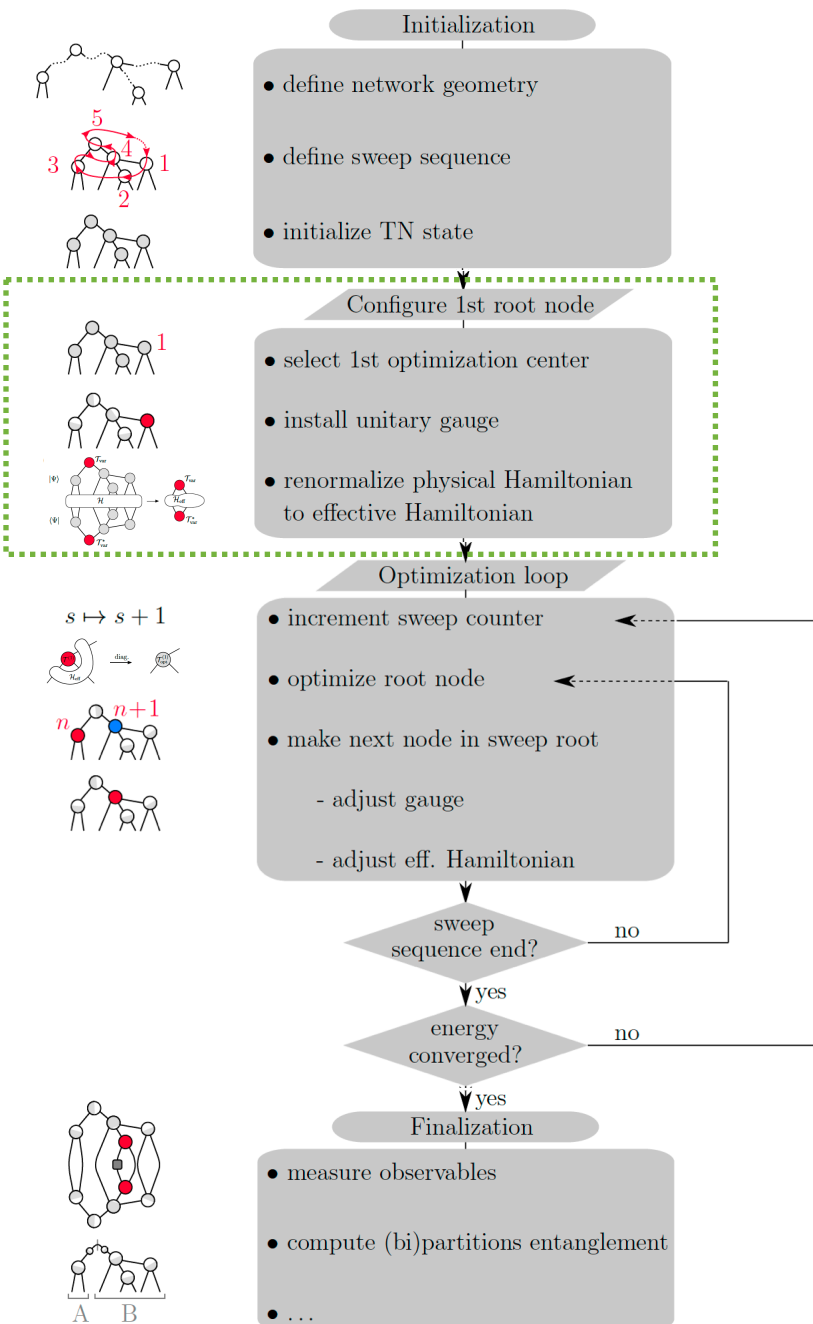


P. Silvi, et al. (MR)
SciPost Phys. Lecture Notes **8** (2019)

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P. Silvi, et al. (MR) SciPost Phys. Lect. Notes 8 (2019)

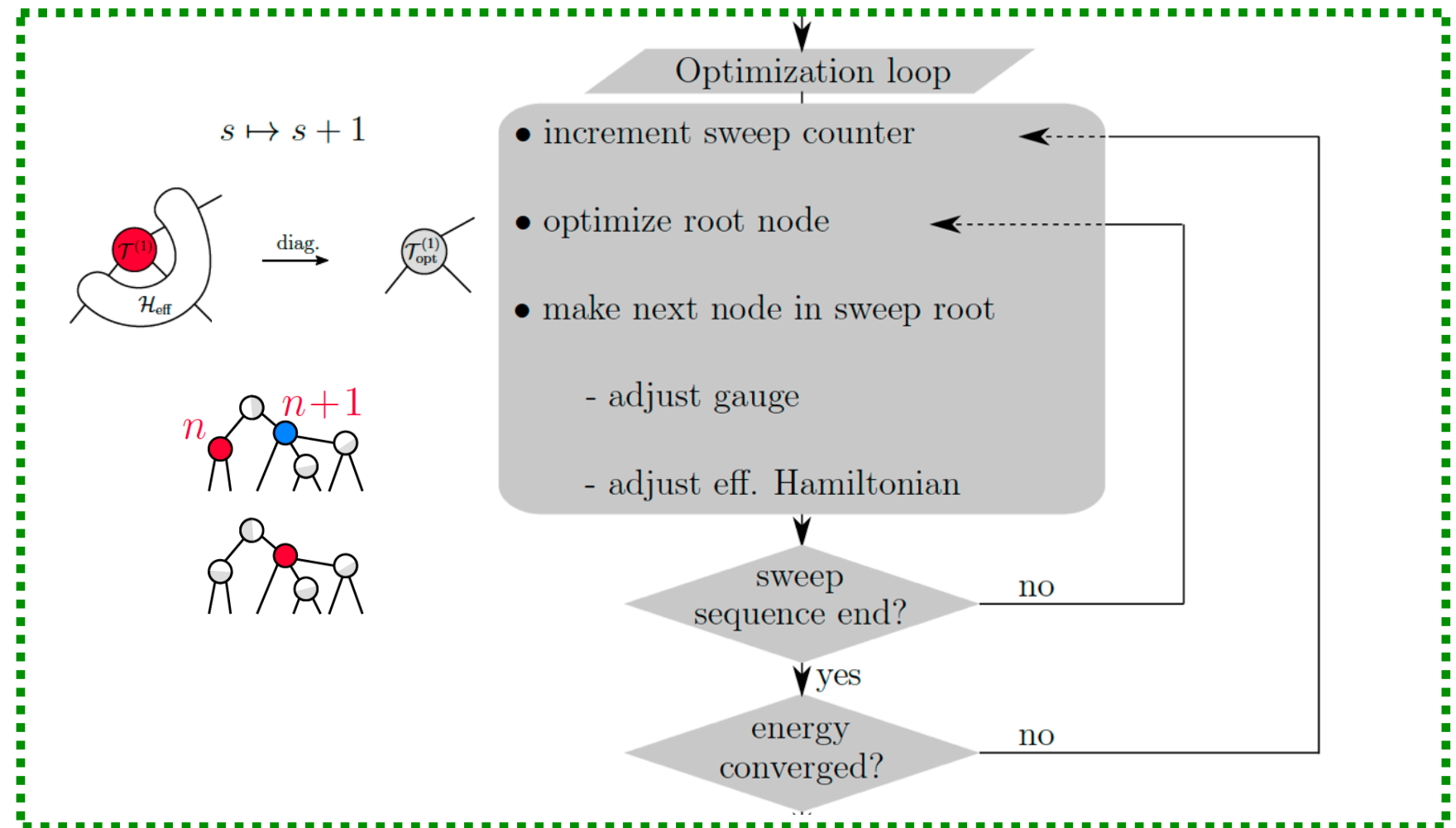
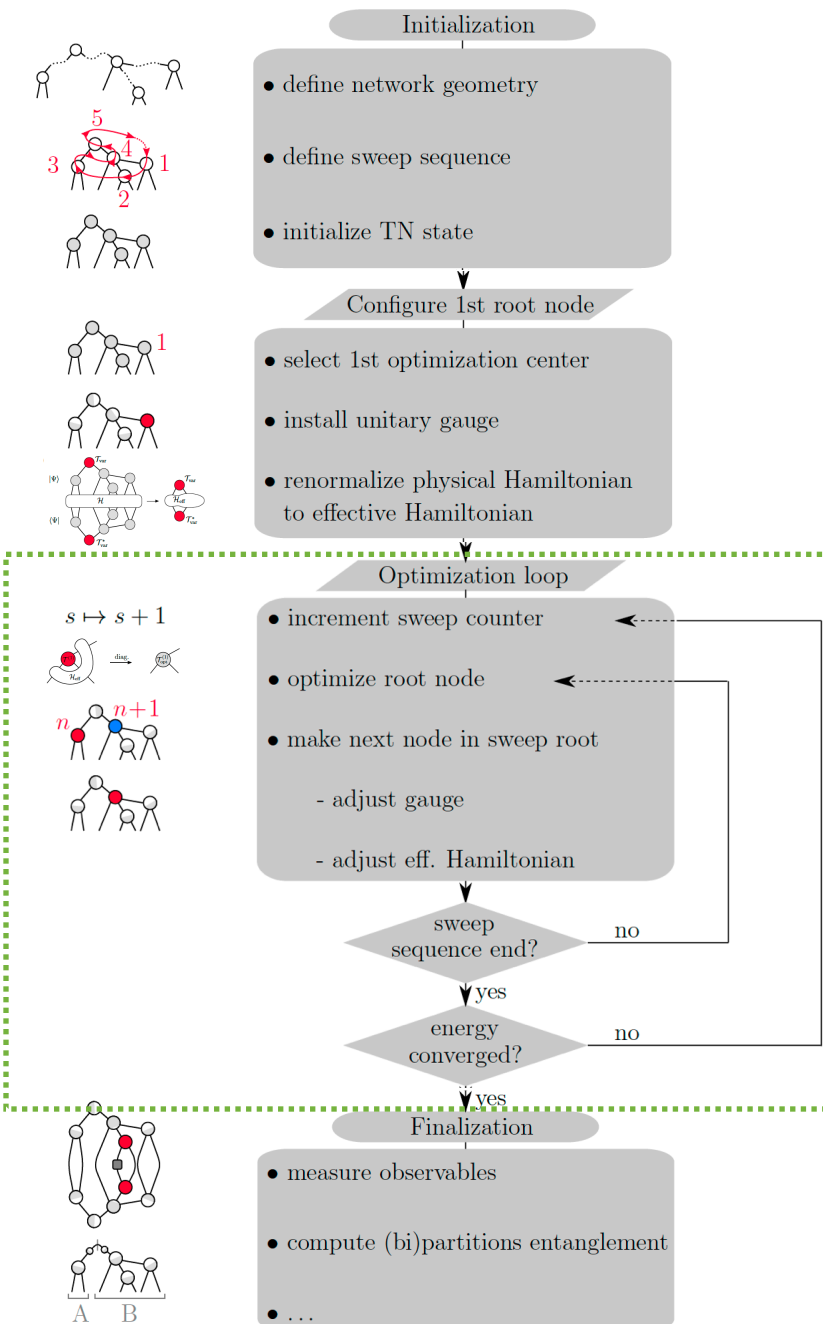


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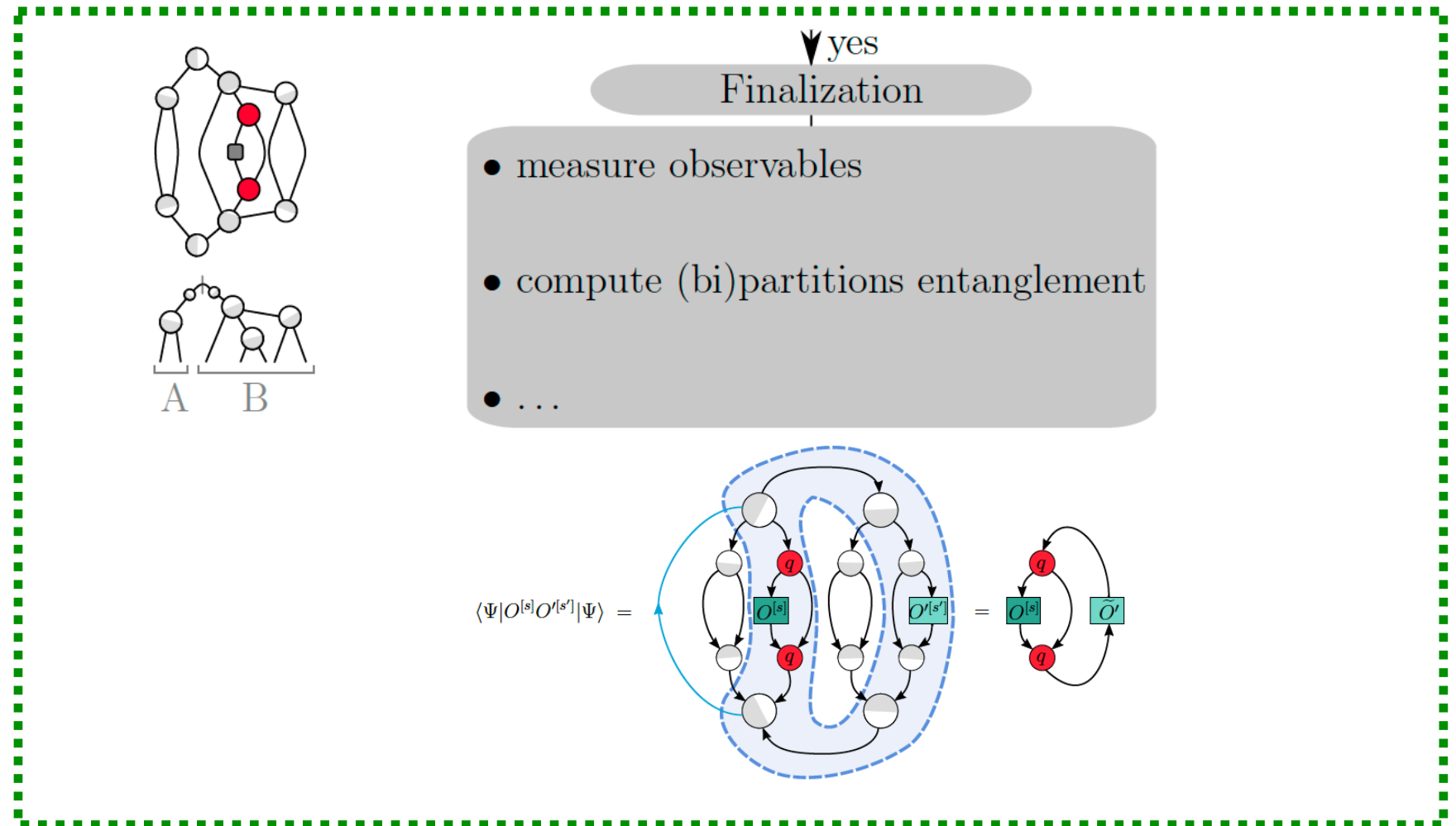
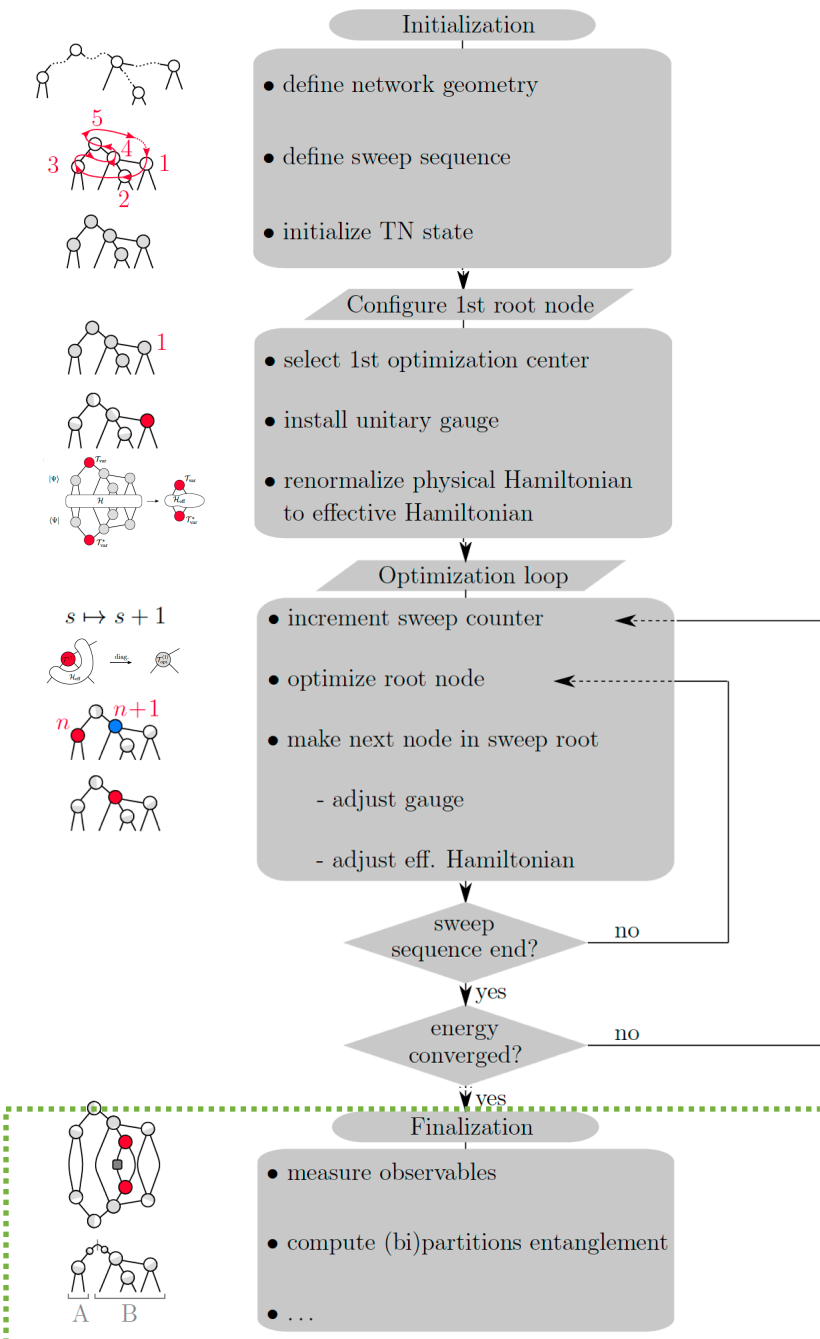


P. Silvi, et al. (MR)
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P. Silvi, et al. (MR) SciPost Phys. Lect. Notes 8 (2019)



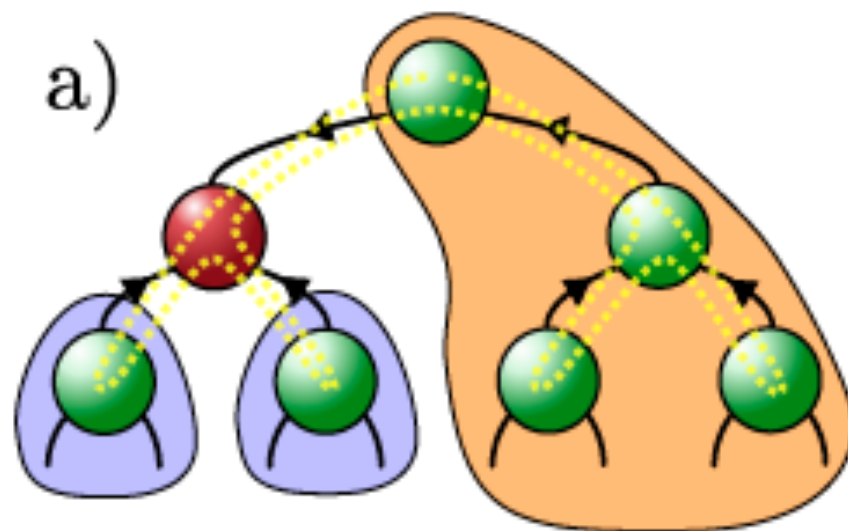
With the same machinery, we can apply TDVP (just like to MPS)

*Haegeman, et al., PRL 107, 070601 ('11) & **PRB 94, 165116 ('16)***

PaECKel et al, Annals of Physics 411, 167998 ('19)

Kloss et al, SciPost Phys. 9, 070 ('20)

$$i\partial_t |\Psi_\theta\rangle = \mathcal{P}_T(\Psi_\theta) \hat{H} |\Psi_\theta\rangle \quad \rightarrow \quad \begin{aligned} i\dot{A}^{[l,i]} &= +H_{\text{eff}}^{[l,i]} A^{[l,i]} \quad (\text{tensors}) \\ i\dot{L}^{[l,i]} &= -H_{\text{eff}}^{[l,i]} L^{[l,i]} \quad (\text{bonds}) \end{aligned}$$



$$\varepsilon \simeq O(\delta t^3)$$

One possible recipe:
sweep the path alternatively
(counter-)clockwise
&
apply (half-)time steps, iff.



TTN: TECHNICAL REMARK ON GPU

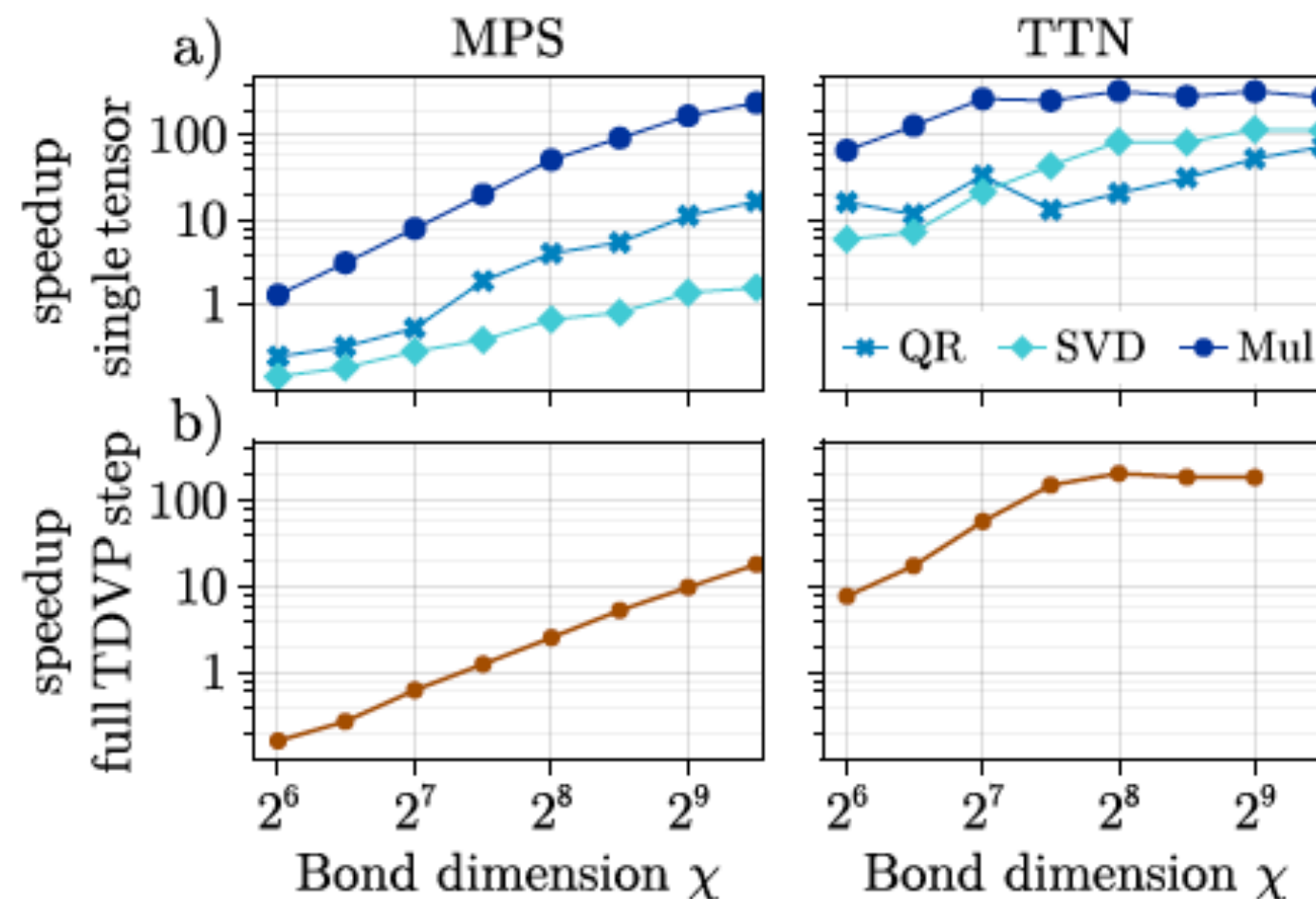
15

W. Krinitsin, et al. ([MR](#)) arXiv:2505.07612 (PRB)

Typical (flattened) tensor size:

$$d \chi \times \chi \quad (\text{MPS})$$
$$\chi^2 \times \chi \quad (\text{TTN}) \text{ tall \& skinny!}$$

As such, they profit at best from GPU acceleration



TREE TENSOR NETWORKS (TTN)

APP 1: STATICS OF (BOSONIC)
FRACTIONAL QUANTUM HALL STATES

HARPER-HOFSTADTER MODEL

17

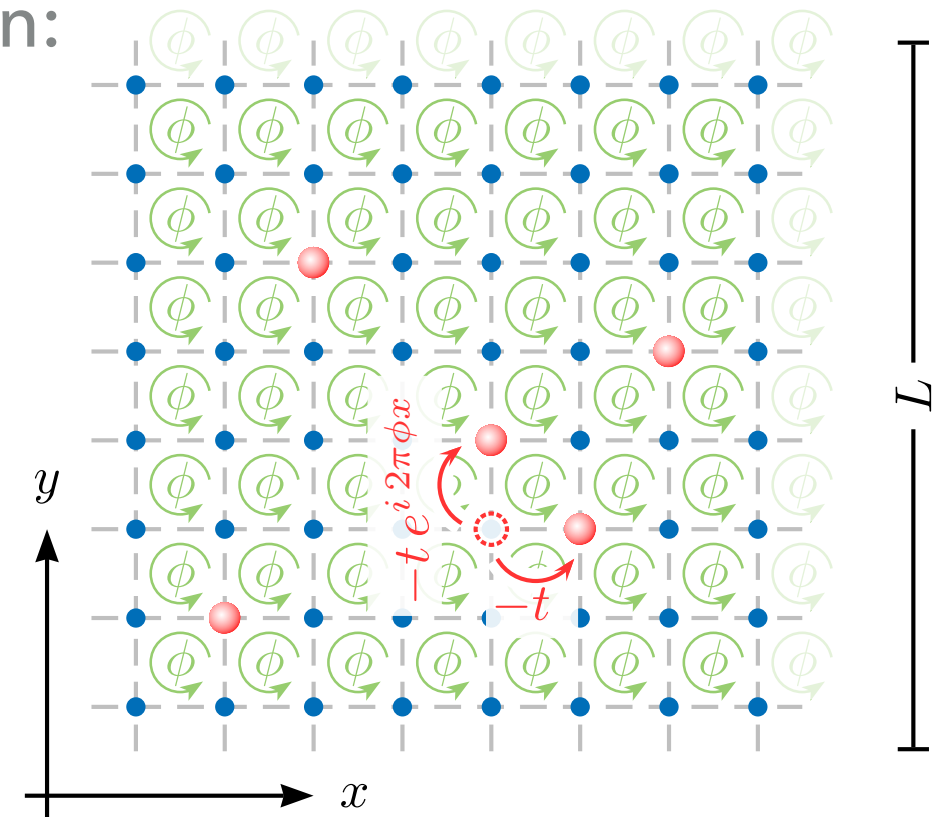
simplest **lattice discretization** of QH Hamiltonian:
Bose-Hubbard model + piercing magnetic flux

three length-scales
in competition
 $\nu = \rho/\alpha$

$$\alpha = \phi/\Phi_0$$

$$a/\ell_B = \sqrt{2\pi\alpha}$$

$$\rho = N/L^2$$



ED works showed large overlap with FQH continuum wavefunctions
even at sizeable finite fluxes, where lattice effects might matter:

Laughlin $\nu = 1/2$

*Sørensen, Demler, Lukin, PRL **94**, 086803 (2005)*

*Hafezi, et al., PRA **76**, 023613 (2007)*

Moore-Read (Pfaffian) $\nu = 1$

*Mazza, **MR**, Lewenstein, Cirac, PRA **82**, 043629 (2010)*

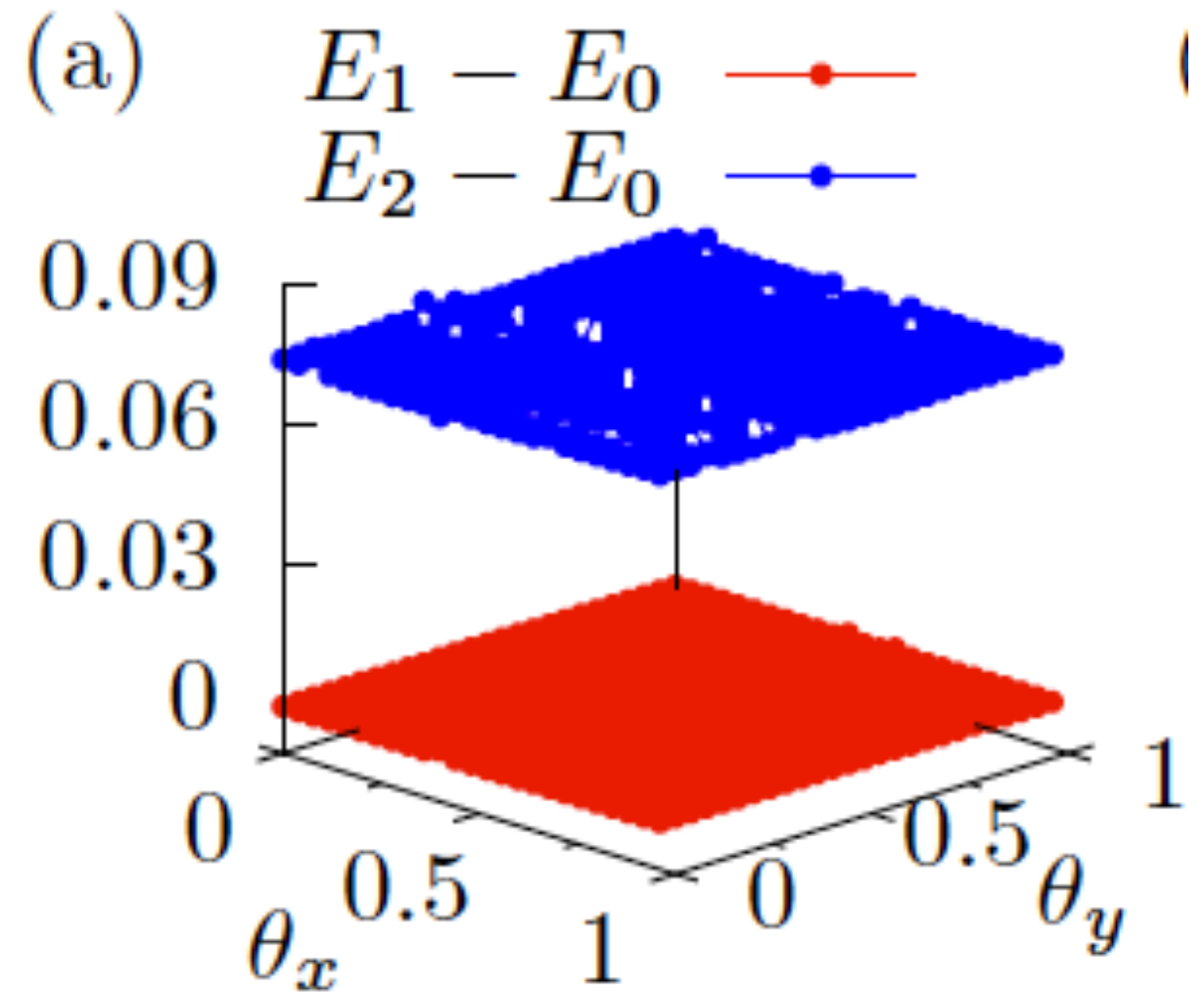
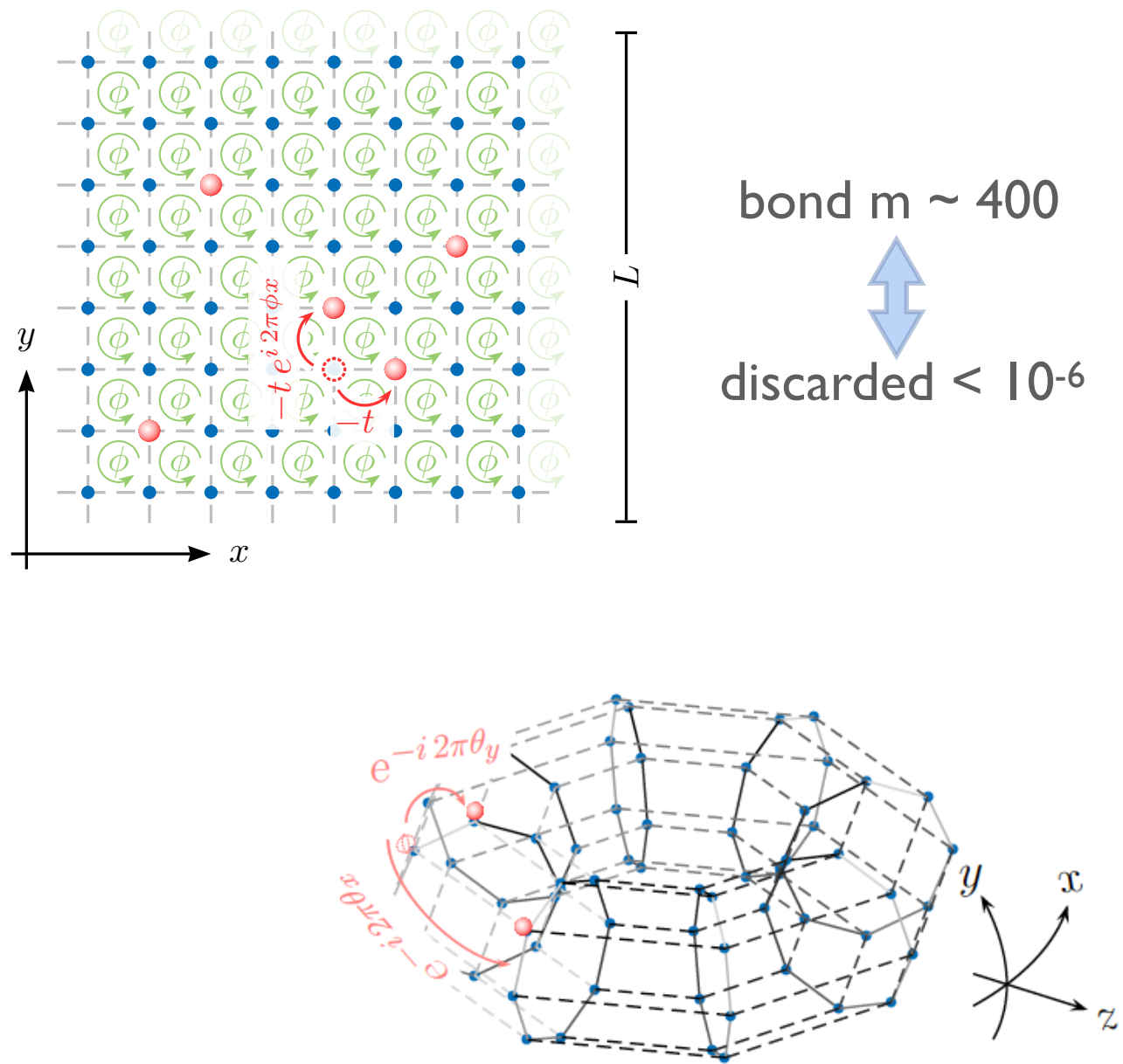
Composite Fermion Theory for Bosons

*Möller, Cooper, PRL **103**, 105303 (2009)*

LOW-ENERGY SPECTRUM VIA TTN

18

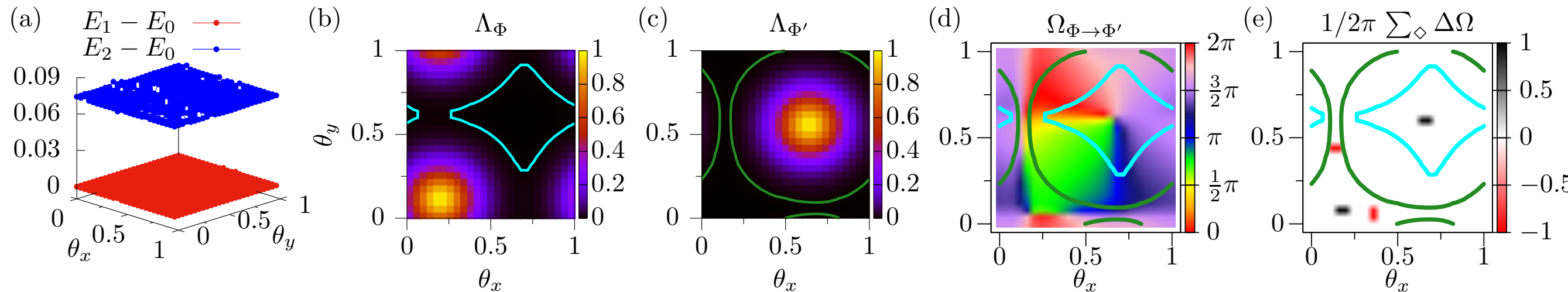
M. Gerster, et al. ([MR](#)) PRB 96, 195123 (2017)



✓ well gapped & doubly degenerate ground manifold for (twisted) PBC

for OBC, instead, $E_2 - E_1 \approx E_1 - E_0$

M. Gerster, et al. ([MR](#)) PRB 96, 195123 (2017)



Discrete procedure based on

Y. Hatsugai, J. Phys. Soc. Jpn. **73**, 2604 (2004) & **74**, 1374 (2005)

Hafezi, Sørensen, Lukin, and Demler, EPL **81**, 10005 (2007)

Mazza, MR, Lewenstein, and Cirac, PRA **82**, 043629 (2010)

(a) check that GS manifold is well-defined at all twists

(b)-(c) choose two reference multiplets & define scalar fields from GS projector P

$$\Lambda_{\Phi^{(i)}} = \det \langle \Phi_j^{(i)} | P(\theta_x, \theta_y) | \Phi_k^{(i)} \rangle$$

(d) if topological, one needs to “patch” the two fields & get a field

$$\Omega_{\Phi \rightarrow \Phi'} = \arg(\det \langle \Phi_j' | P(\theta_x, \theta_y) | \Phi_k \rangle)$$

(e) count the branch cuts (with sign) in either of the two regions

$$\checkmark \text{ MBCN} / \text{deg(GS)} = 1/2 !$$

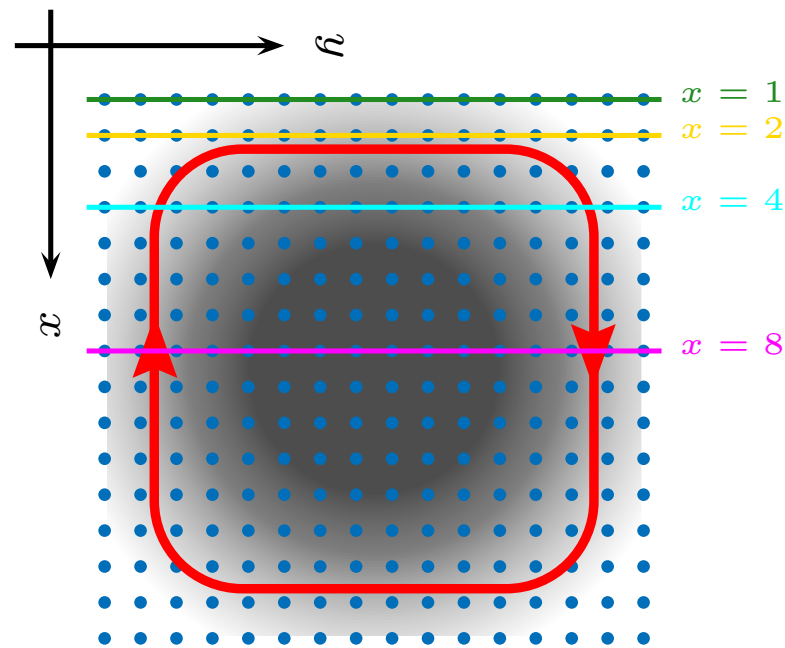
CORRELATION FUNCTIONS & EDGE MODES

20

M. Gerster, et al. ([MR](#)) PRB 96, 195123 (2017)

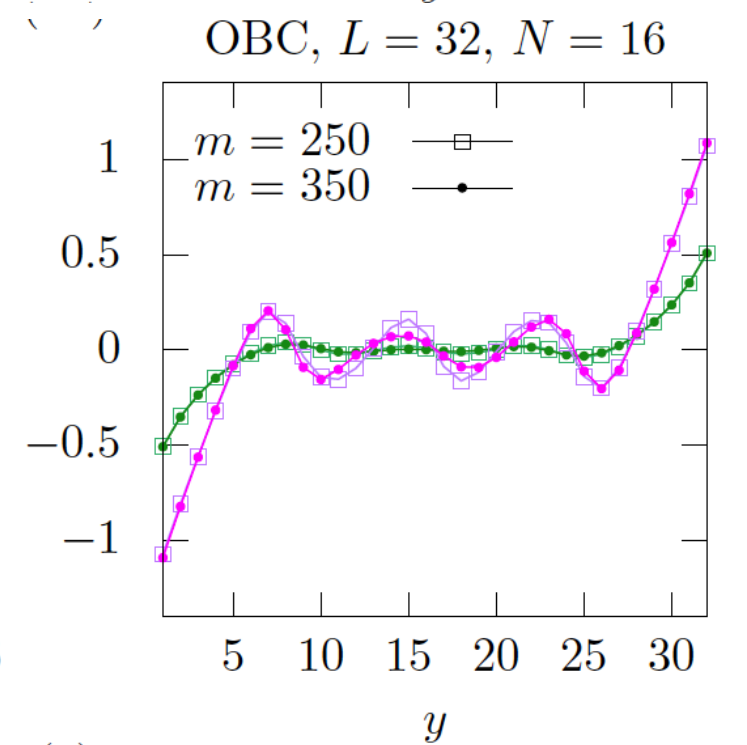
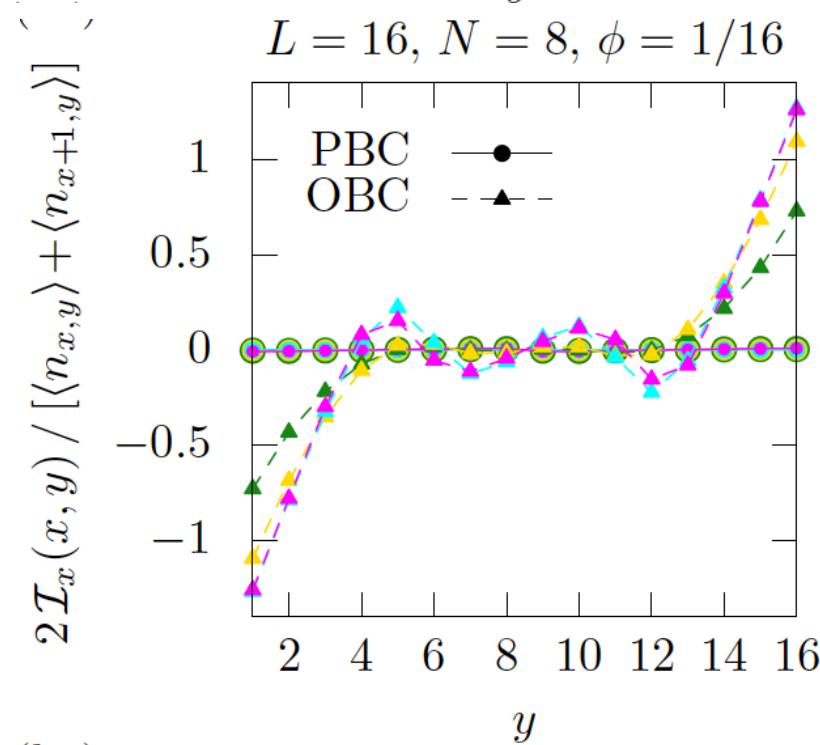
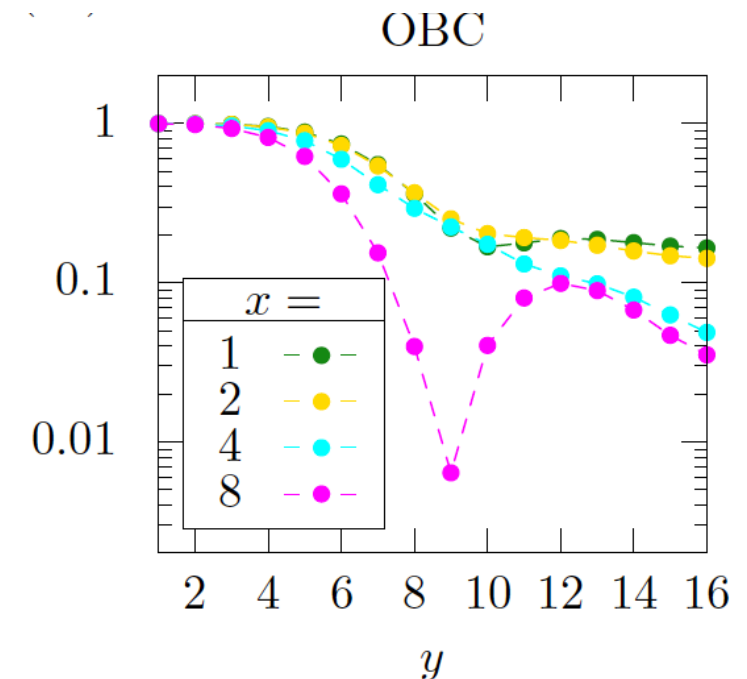
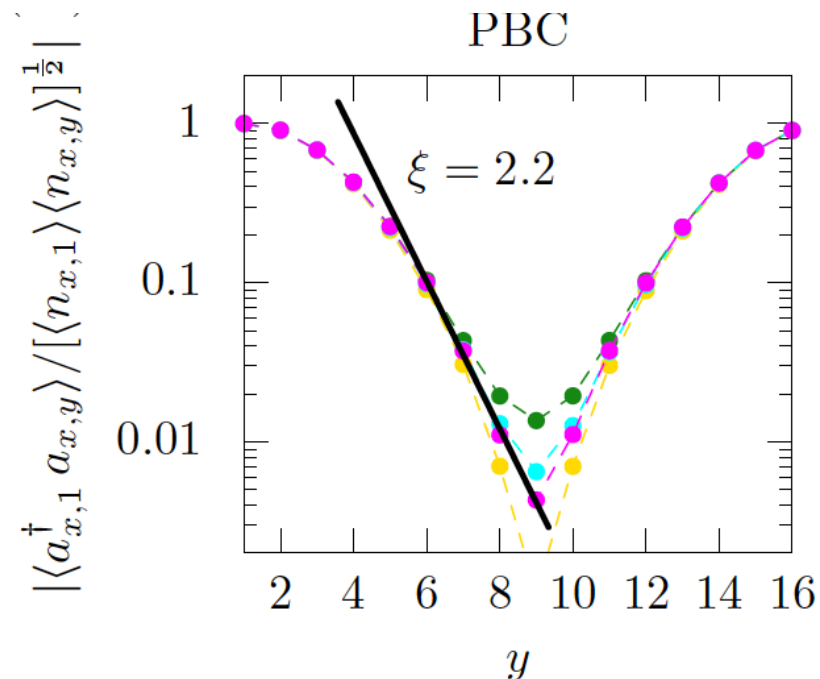
Green functions

$$\mathcal{G}(x, x'; y, y') = \langle a_{x,y}^\dagger a_{x',y'} \rangle$$

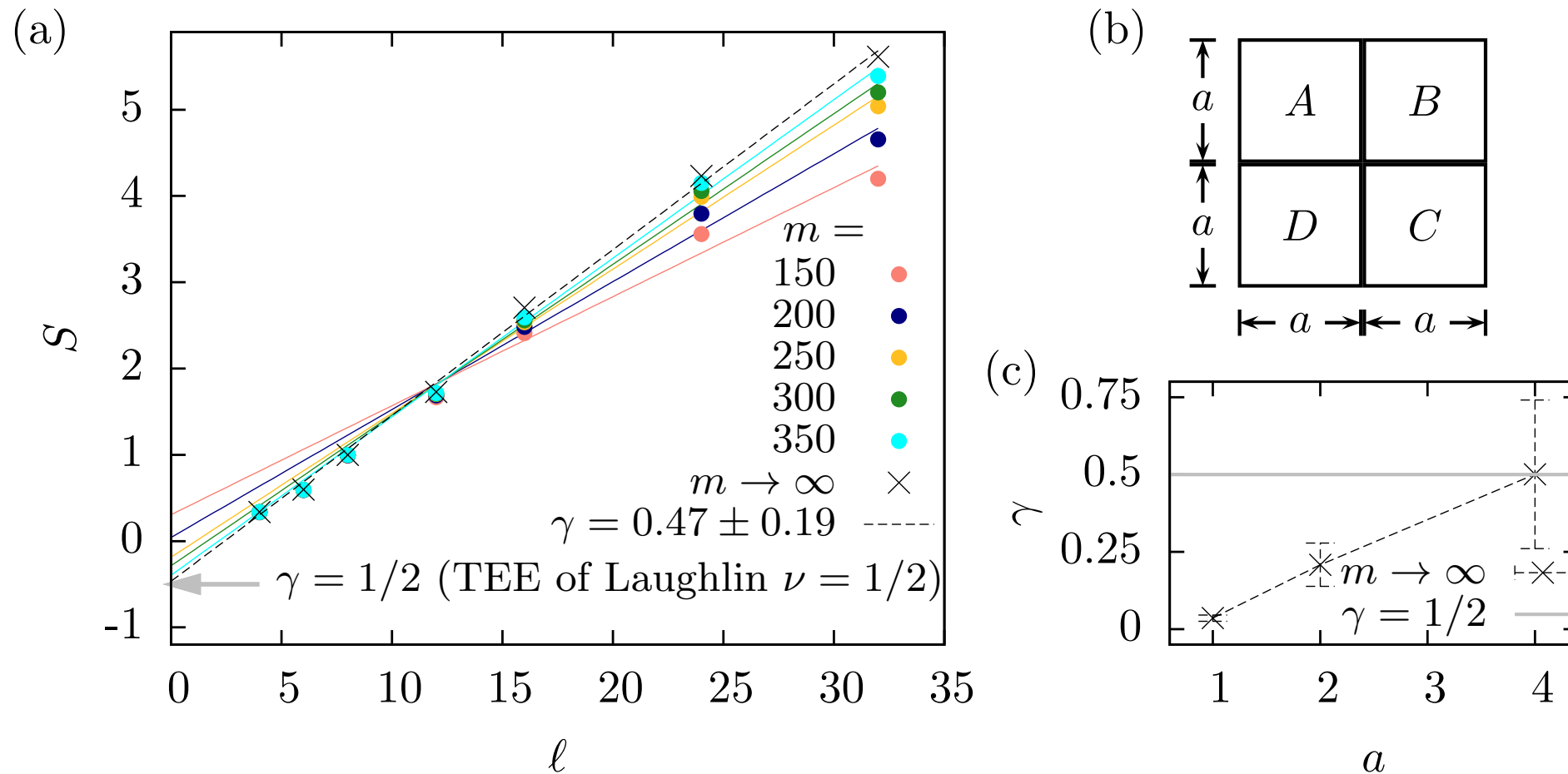


Local currents

$$\mathcal{I}_x(x, y) = i \langle a_{x+1,y}^\dagger a_{x,y} - a_{x+1,y} a_{x,y}^\dagger \rangle$$



✓ observables consistent with gapped bulk & gapless edge modes!



Topological correction to area law: $S(\ell) = c\ell - \gamma$

A. Kitaev and J. Preskill, PRL **96**, 110404 (2006)

M. Levin and X.-G. Wen, PRL **96**, 110405 (2006)

$$-\gamma = S_A + S_B + S_C + S_D - S_{AB} - S_{BC} - S_{CD} - S_{DA} + S_{ABCD}$$

related to “quantum dimension” of anyonic excitations: Laughlin $\nu = 1/q \implies \gamma = \log_2 \sqrt{q}$

* consistency with topological predictions (though not conclusive)

DENSITY PROFILES & QUASI-HOLES

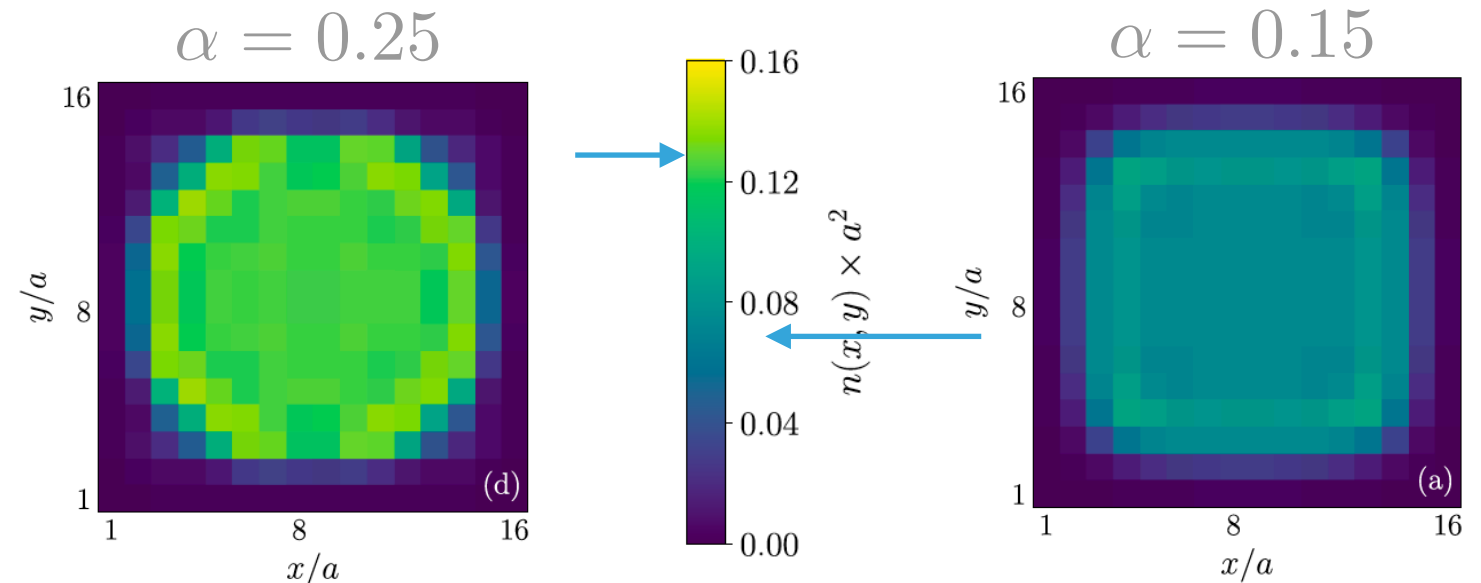
22

E. Macaluso, et al. (MR) PRR **2**, 013145 (2020)

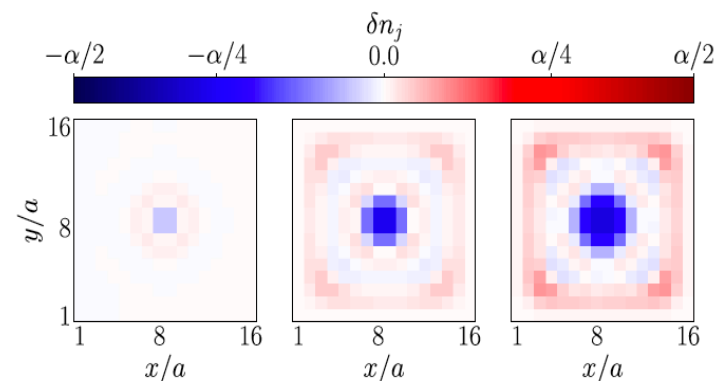
✓ Harmonic confinement:
flat incompressible profile

+ pinning potential
to generate quasi-holes

✓ Charge & braiding statistics from depletion profiles

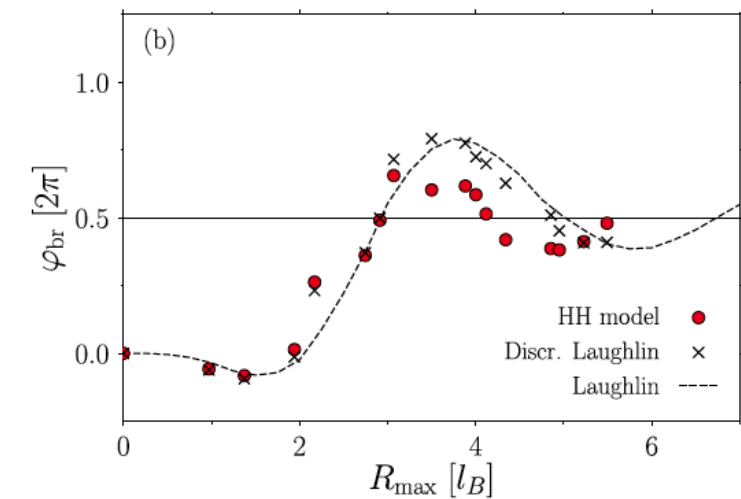
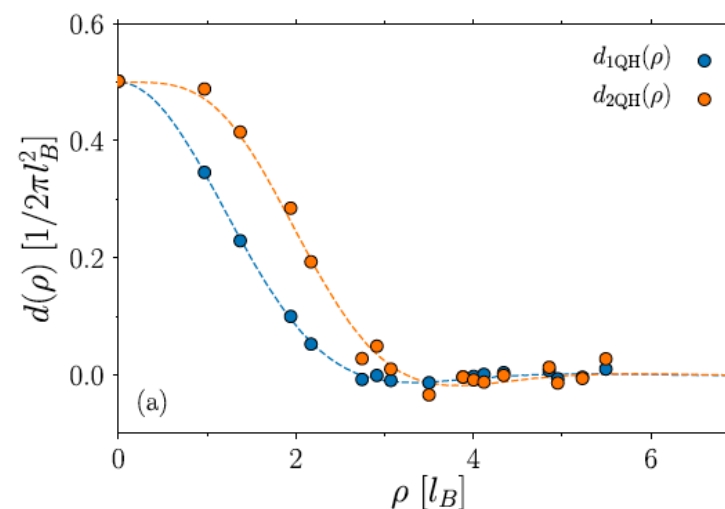
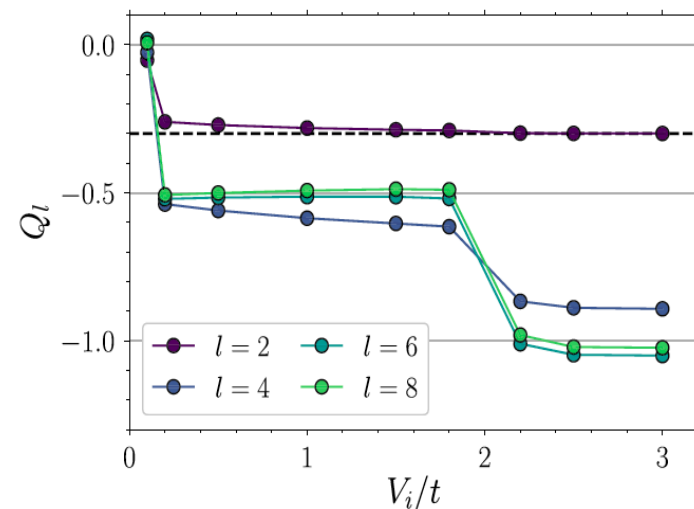


E. Macaluso, et al.,
PRL **123**, 266801 (2019)



$$Q_l = \int d\vec{\rho} d(\vec{\rho}) \equiv \int d\vec{\rho} [n(\vec{\rho}) - n_{0\text{QH}}(\vec{\rho})]$$

$$\frac{\varphi_{\text{br}}}{2\pi} = \frac{1}{2l_B^2} \int d\vec{\rho} \rho^2 [d_{2\text{QH}}(\vec{\rho}) - 2d_{1\text{QH}}(\vec{\rho})]$$



TREE TENSOR NETWORKS (TTN)

APP 2: DYNAMICS OF 2D ISING MODELS

W. Krinitsin et al. (MR), PRL 134, 240402 ('25)

W. Krinitsin, et al. (MR) arXiv:2505.07612 (PRB)

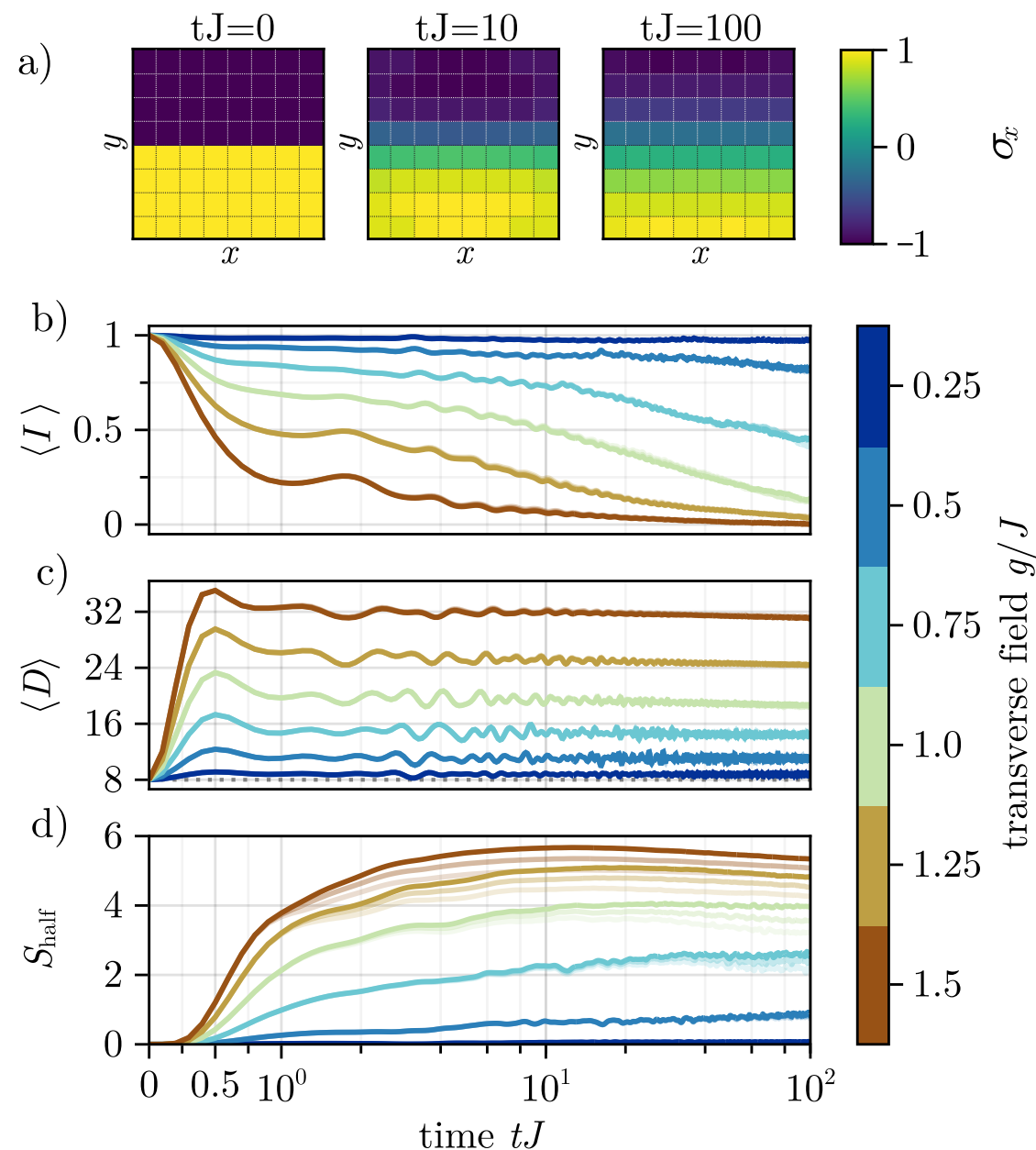


INTERFACES IN THE 2D ISING MODEL

24

W. Krinitsin et al. (MR), PRL 134, 240402 ('25)

$$H_{\text{TFI}} = -J \sum_{\langle j,k \rangle} \sigma_j^x \sigma_k^x - g \sum_j \sigma_j^z$$



Initial domain-wall
+
quench in transverse field

TTN-2D & TDVP

Imbalance $\langle I \rangle = \langle \sigma^x \rangle_{\text{top}} - \langle \sigma^x \rangle_{\text{bottom}}$

Domain-walls $D = \sum \langle j, j' \rangle (1 - \sigma_j^x \sigma_{j'}^x)$

Entanglement entropy (sanity check)

! Long-lived non-thermal plateaus up to $g/J \sim 1$?

W. Krinitsin et al. (MR), PRL 134, 240402 ('25)

$$H_{\text{TFI}} = -J \sum_{\langle j,k \rangle} \sigma_j^x \sigma_k^x - g \sum_j \sigma_j^z$$

! Long-lived non-thermal plateaus up to $g/J \sim 1$?

$g/J \ll 1$: effective PXP model $\rightarrow$ Hilbert-space fragmentation

$$H_{\text{PXP}} = - \sum_i h \sigma_i^z + g \left(|\ulcorner\rangle_i \langle \llcorner| + |\llcorner\rangle_i \langle \urcorner| + |\sqcap\rangle_i \langle \sqcup| + \text{h.c.} \right)$$

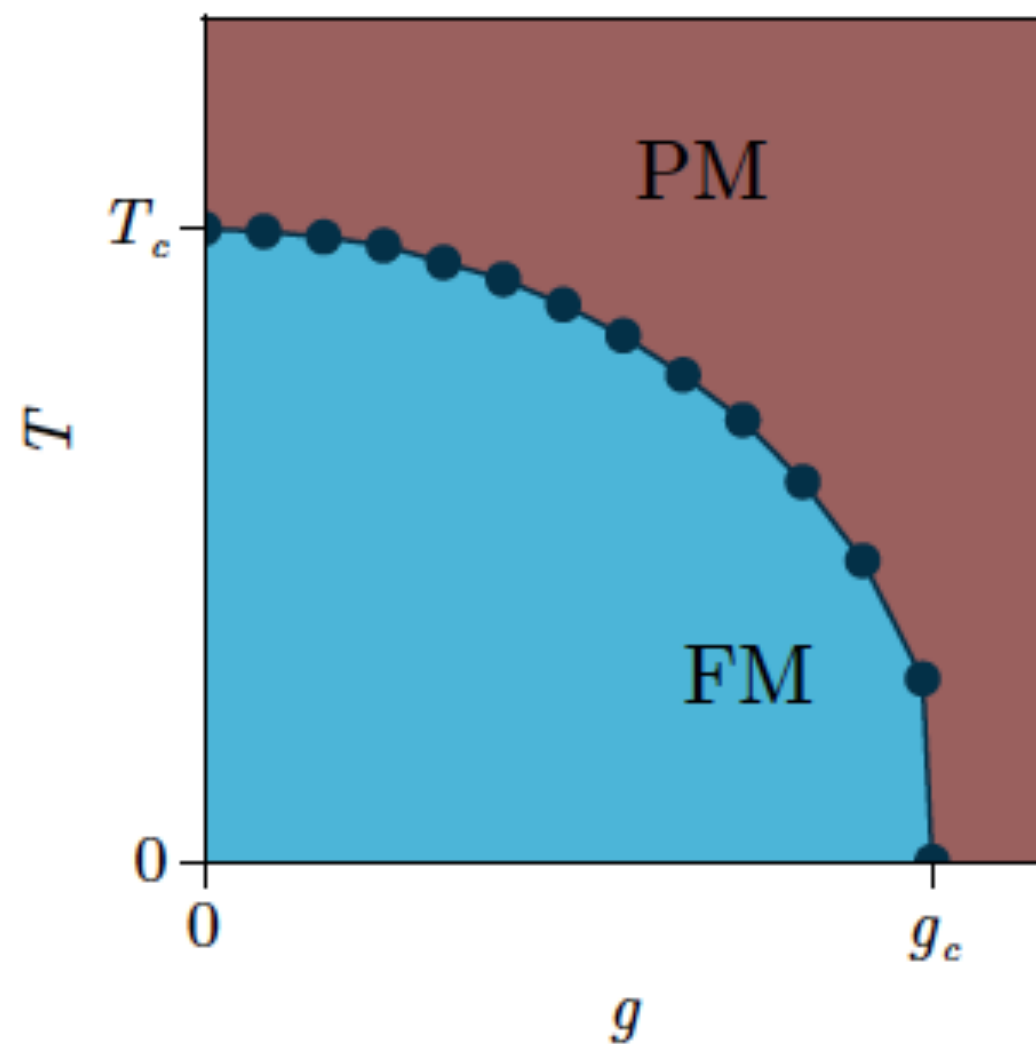
F. Balducci, et al., PRL 129, 120601 ('22) & PRB 107, 024306 ('23)

L. Pavesic, et al, PRB 111, L140305 ('25) also TTN :-)

BUT *here* we have too long domain walls to be perturbative!

$$H_{\text{TFI}} = -J \sum_{\langle j,k \rangle} \sigma_j^x \sigma_k^x - g \sum_j \sigma_j^z$$

! Long-lived non-thermal plateaus up to $g/J \sim 1$?



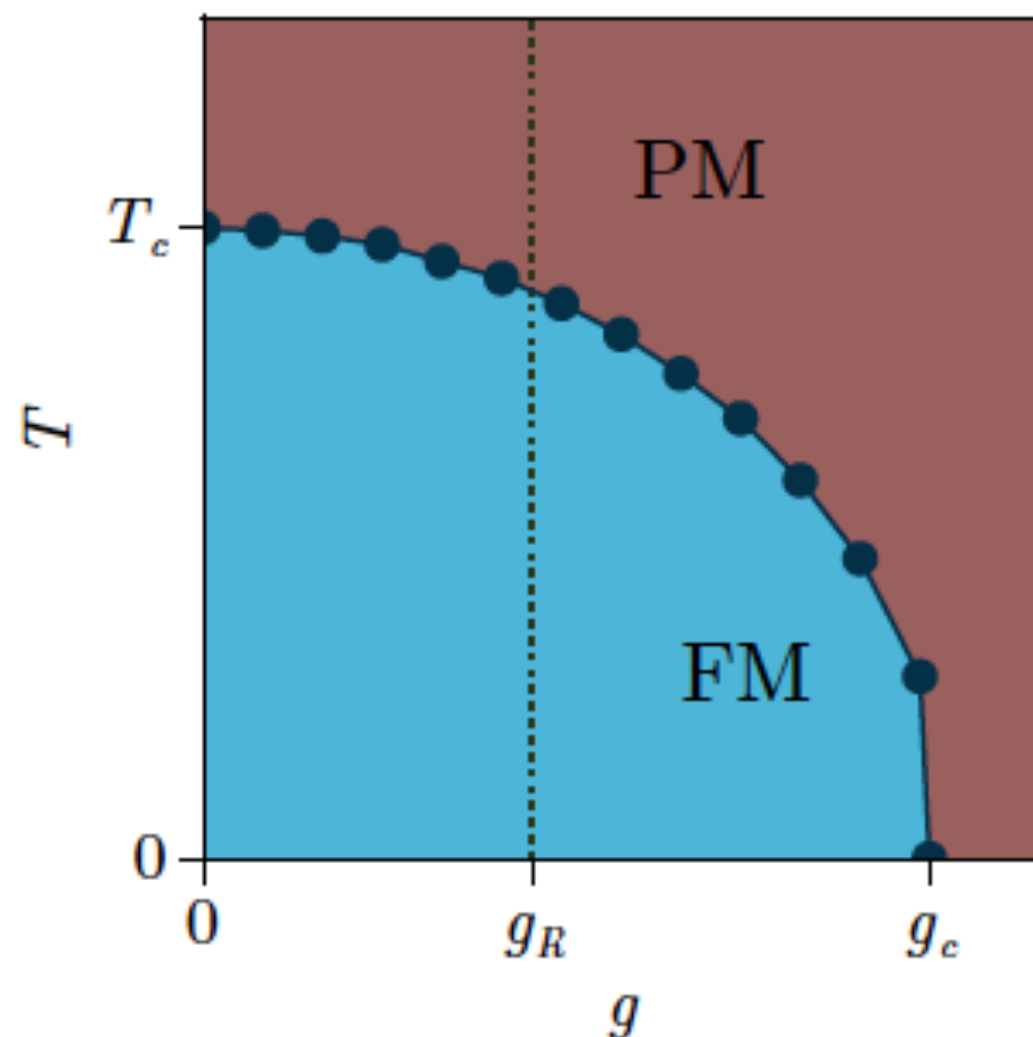
Quantum phase transition at $(g_c, 0)$

Classical phase transition at $(0, T_c)$

W. Krinitsin et al. (MR), PRL 134, 240402 ('25)

$$H_{\text{TFI}} = -J \sum_{\langle j,k \rangle} \sigma_j^x \sigma_k^x - g \sum_j \sigma_j^z$$

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Quantum phase transition at $(g_c, 0)$

Classical phase transition at $(0, T_c)$

Quantum roughening transition at $(g_R, 0)$

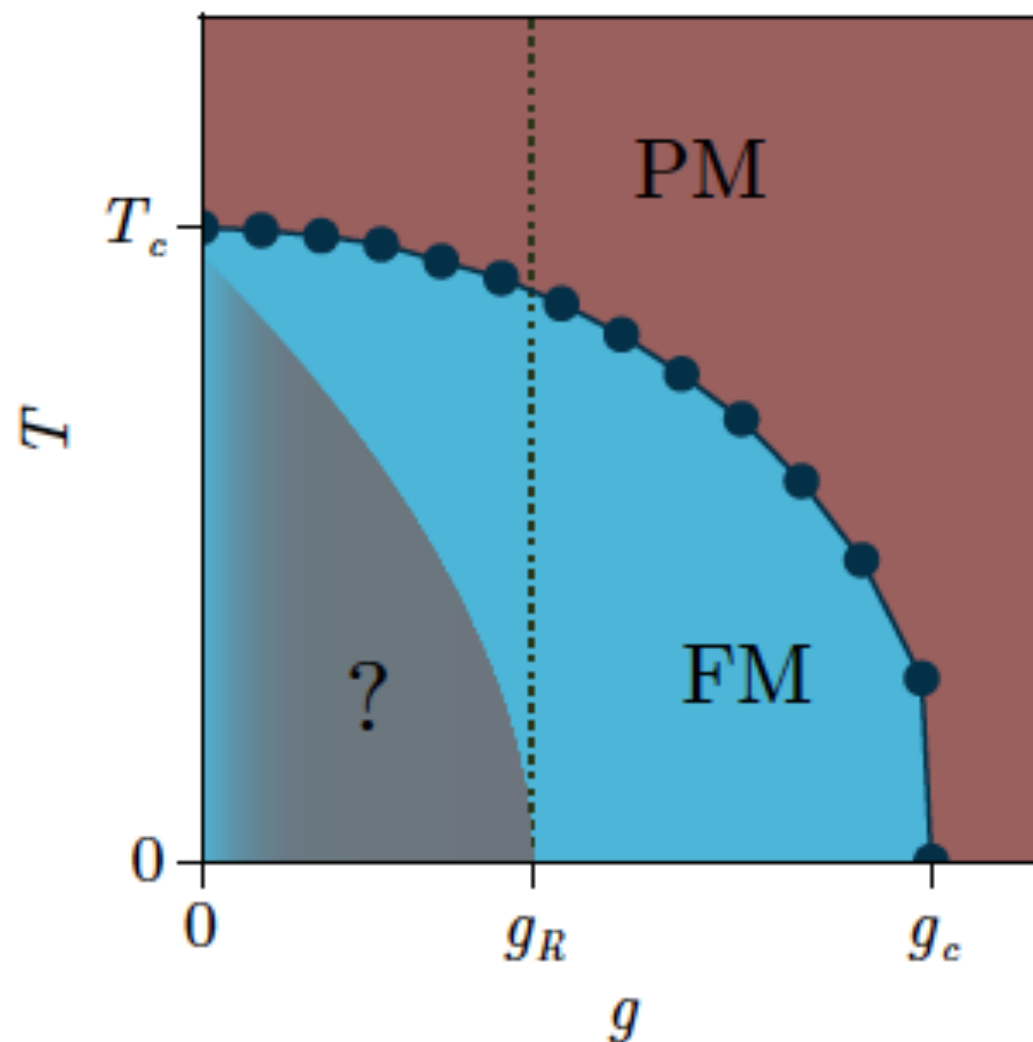
Hasenfratz Nucl. Phys. B 180, 353 (1981)

Fradkin PRB 28, 5338 (1983)

W. Krinitsin et al. (MR), PRL 134, 240402 ('25)

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Quantum phase transition at $(g_c, 0)$

Classical phase transition at $(0, T_c)$

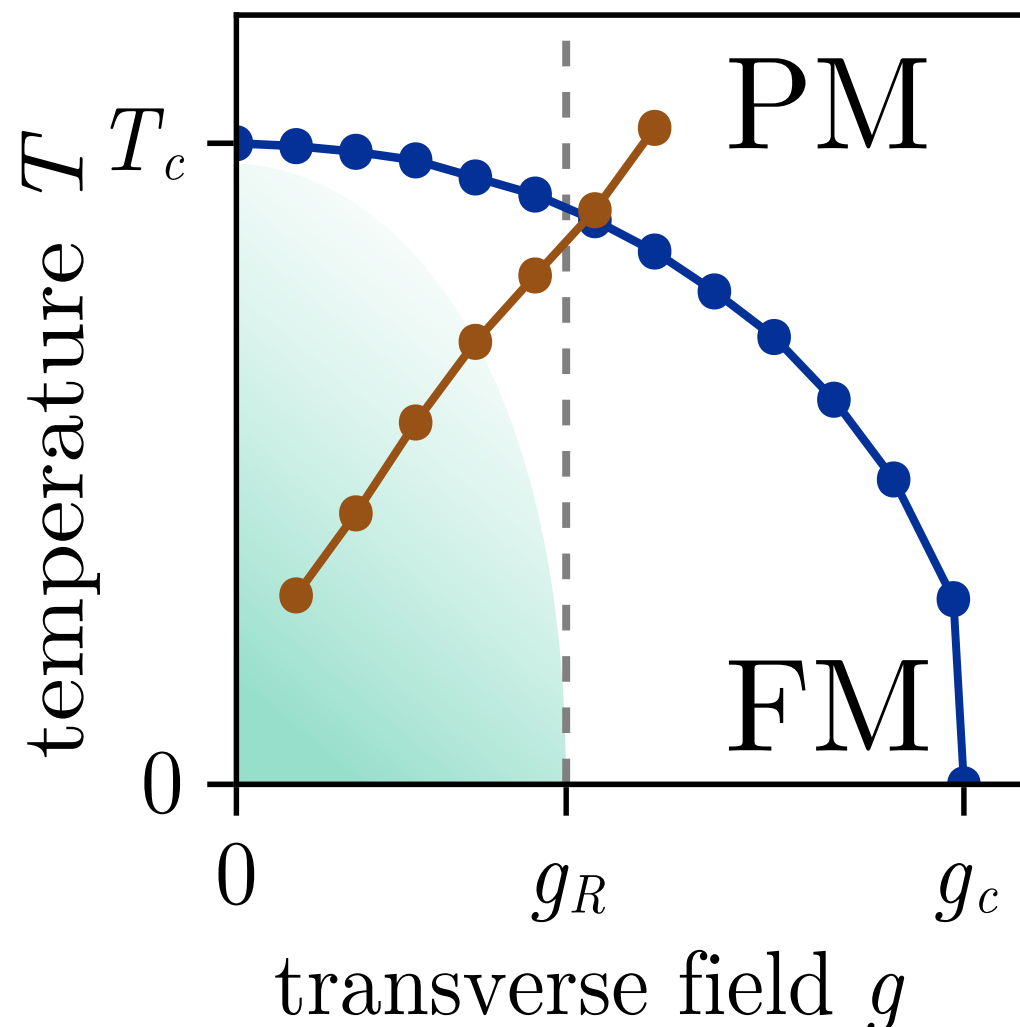
Quantum roughening transition at $(g_R, 0)$
(actually a BKT) Hasenfratz Nucl. Phys. B 180, 353 (1981)
Fradkin PRB 28, 5338 (1983)

No classical roughening in 2D, but ...
... finite-size effects are very persistent!

W. Krinitsin et al. (MR), PRL 134, 240402 ('25)

$$H_{\text{TFI}} = -J \sum_{\langle j,k \rangle} \sigma_j^x \sigma_k^x - g \sum_j \sigma_j^z$$

! Long-lived non-thermal plateaus up to $g/J \sim 1$?



Quantum phase transition at $(g_c, 0)$

Classical phase transition at $(0, T_c)$

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Fradkin PRB 28, 5338 (1983)

No classical roughening in 2D, but ...
... finite-size effects are very persistent!

Effective temperature via quench energy

W. Krinitsin et al. (MR), PRL 134, 240402 ('25)

$$H_{\text{TFI}} = -J \sum_{\langle j,k \rangle} \sigma_j^x \sigma_k^x - g \sum_j \sigma_j^z$$

Quantum roughening transition at $(g_R, 0)$ (actually a BKT)

Hasenfratz3 Nucl. Phys. B 180, 353 (1981)

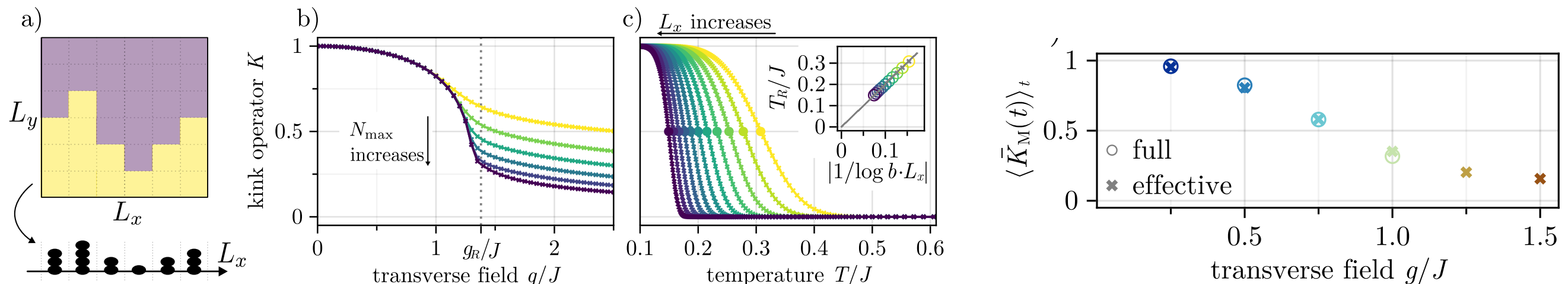
Fradkin PRB 28, 5338 (1983)

No classical roughening in 2D, but ... finite-size effects are very persistent!

No overhangs, no bubbles ==> **Solid-On-Solid** effective model

$$H_{\text{eff}} = \sum_j \left[2J |N_{j+1} - N_j| - g(E_j + E_j^\dagger) \right]$$

$$K_M(\alpha, l) = \cos(\alpha(N_1 - N_l)) \quad \lim_{\alpha \rightarrow 0} \lim_{l \rightarrow \infty} K_M(\alpha, l) = \begin{cases} \text{const} & g < g_R \\ 0 & g > g_R \end{cases}$$



TREE TENSOR NETWORKS (TTN)

OTHER APPLICATIONS & EXTENSIONS

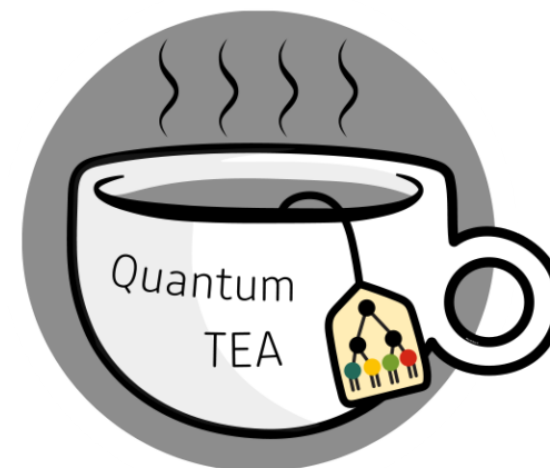
TREE TENSOR NETWORKS (TTN)

OTHER APPLICATIONS & EXTENSIONS

Install Quantum TEA

Select values to generate command

Application	<input checked="" type="button" value="Schrödinger/Lindblad"/>	Quantum circuits	Machine learning
Tensor library	<input checked="" type="button" value="Numpy"/>	PyTorch	
Symmetric tensors	Enabled	<input checked="" type="button" value="Disabled"/>	
Installation type	<input checked="" type="button" value="Pip"/>	Source code	



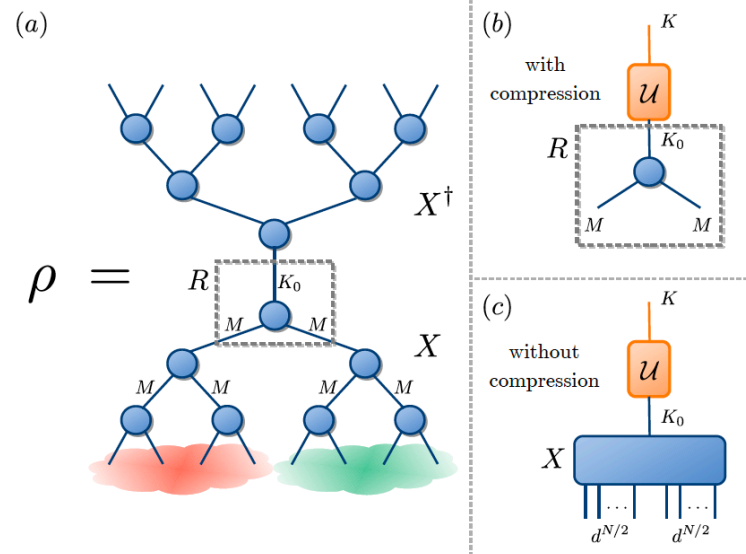
www.quantumtea.it



S. Montangero's group
(Uni Padua, IT)

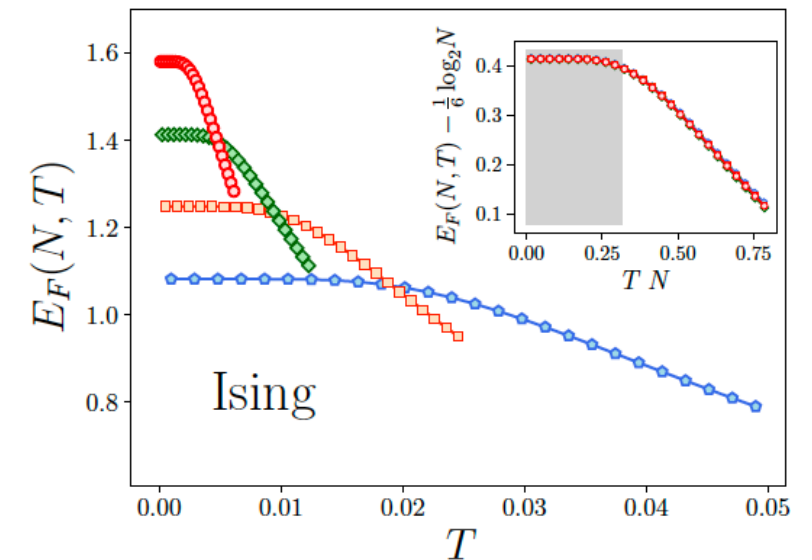
(SOME) OTHER APPLICATIONS OF TTN

29



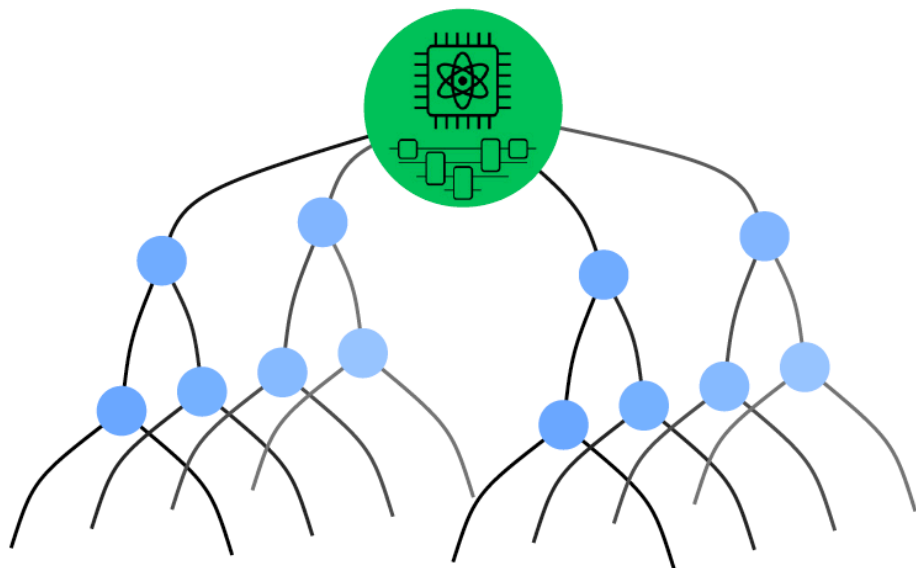
Entanglement of formation of mixed many-body states via Tree Tensor Operators

Arceci, Silvi, Montangelo, PRL 128, 040501 ('22)



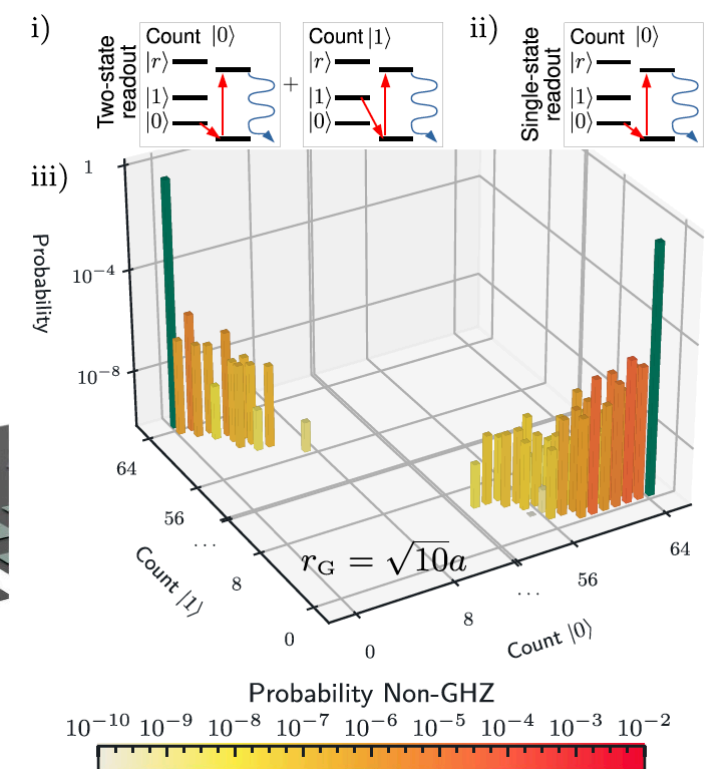
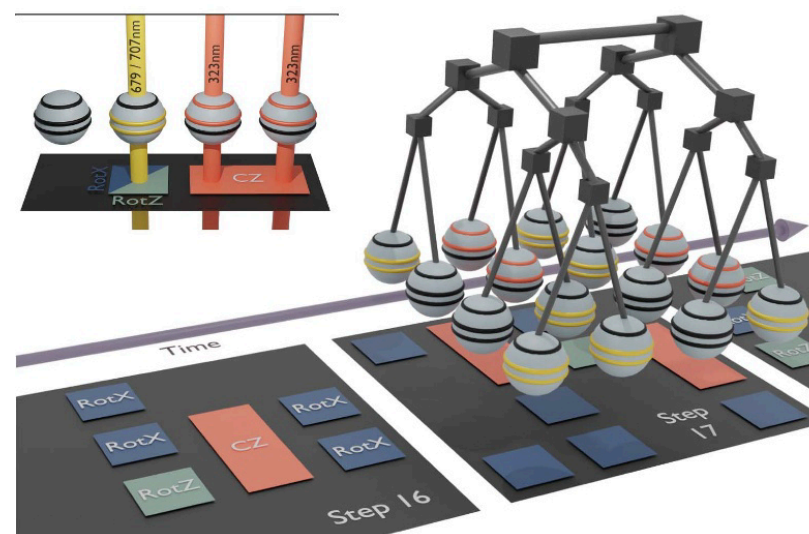
Hybrid Tree Tensor Networks for Quantum Simulation

Schuhmacher, et al., PRX QUANTUM 6, 010320 (2025)



Ab-initio TTN digital twin of Rydberg-atom quantum computer

Jaschke, et al., Quantum Sci. & Tech. 9, 035055 ('24)

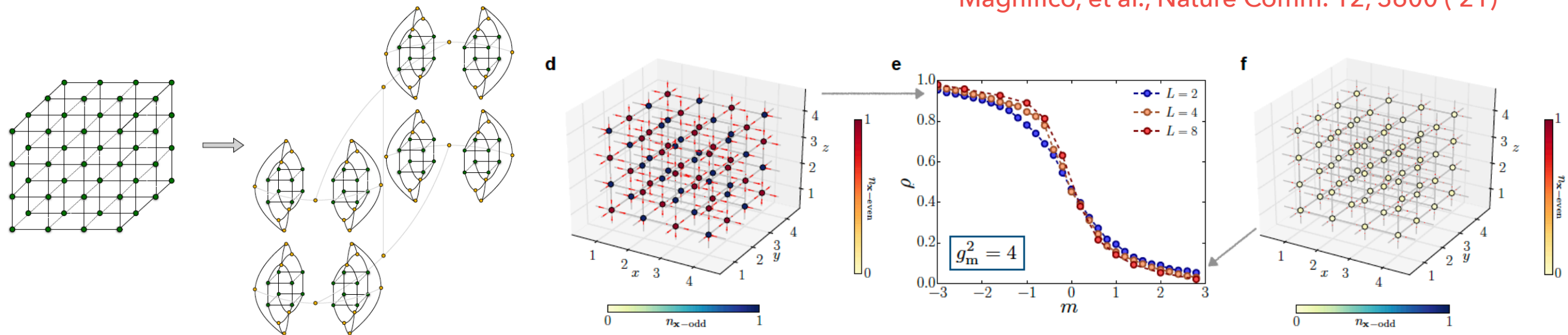


(SOME) OTHER APPLICATIONS OF TTN

30

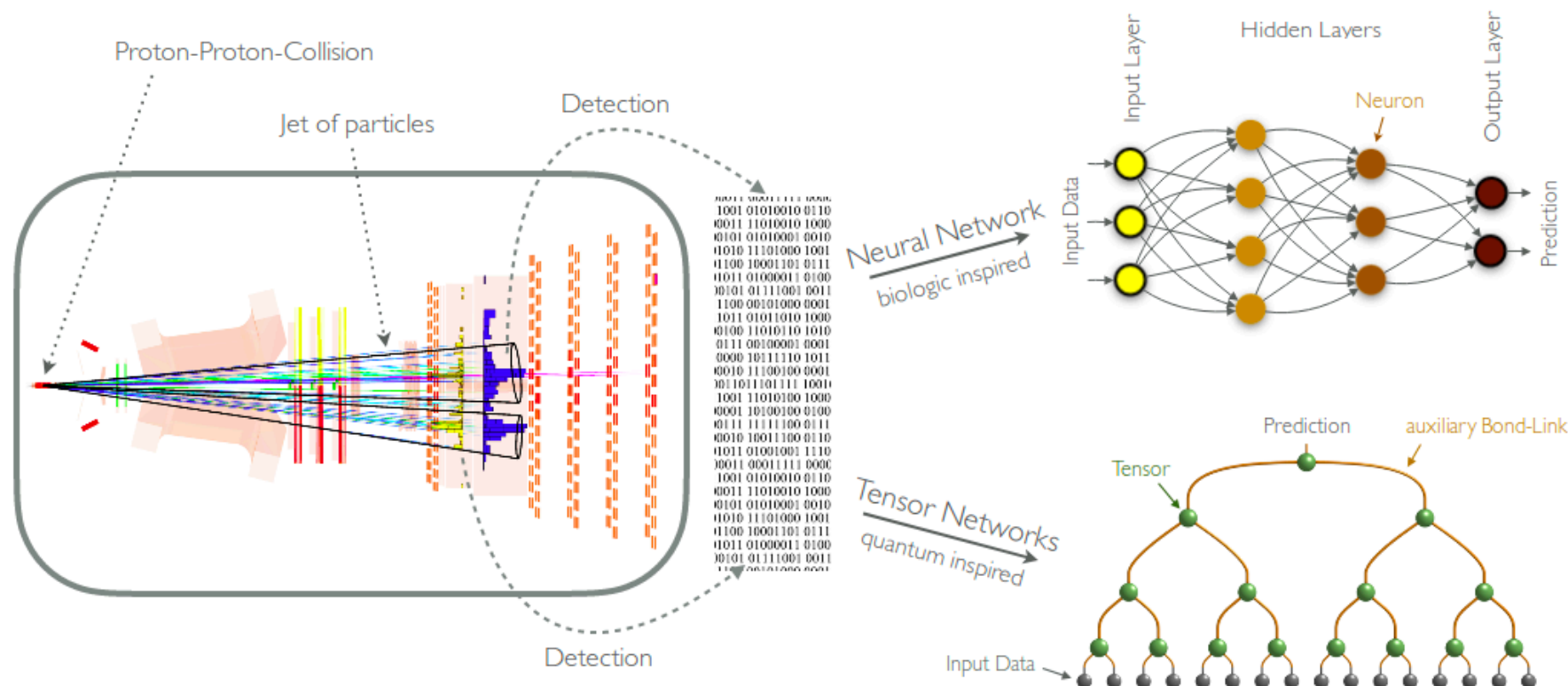
Lattice Quantum Electrodynamics in (3+1)-dimensions at finite density

Magnifico, et al., Nature Comm. 12, 3600 ('21)



Quantum-inspired Machine Learning on high-energy physics data (LHCb)

Felser, et al., npj Quantum Info. 7, 111 ('21)

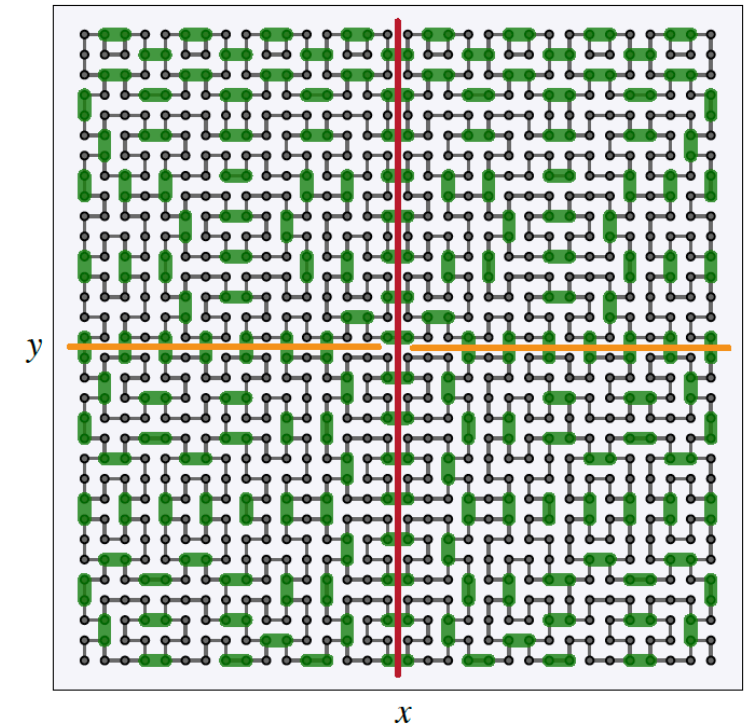
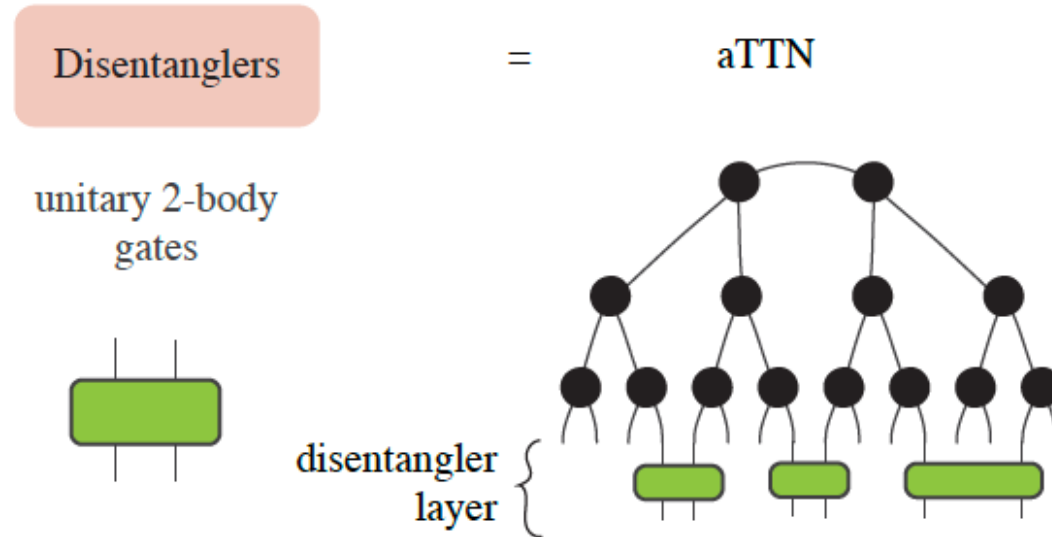


AUGMENTED TTN: EVEN BETTER?

31

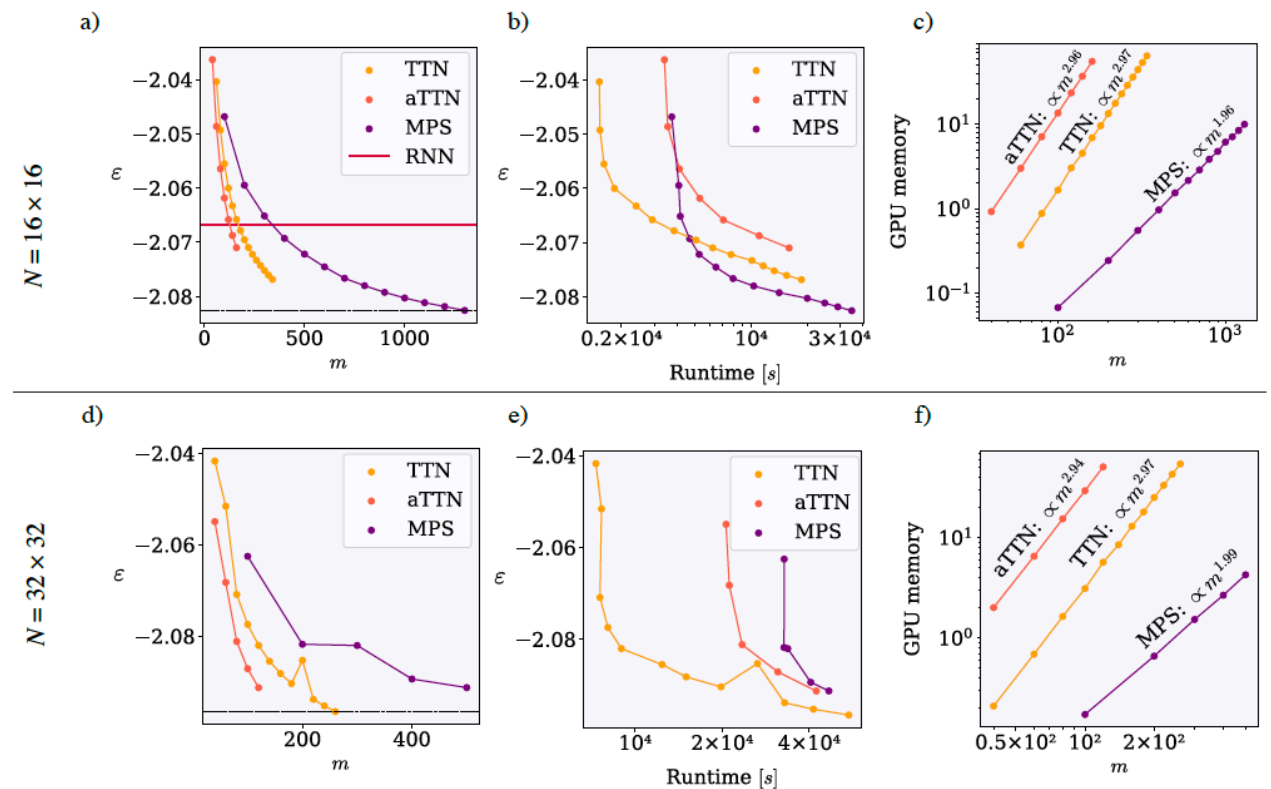
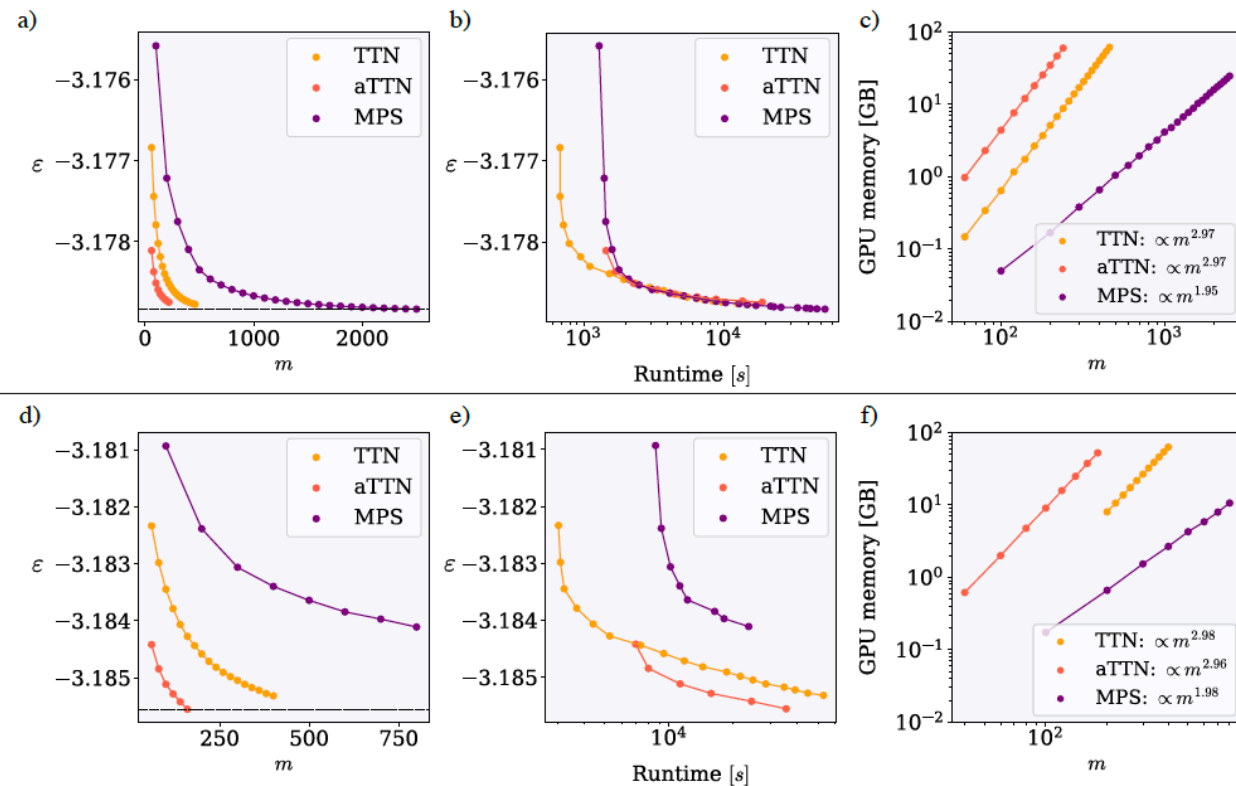
Felser, et al., PRL 126, 170603 ('21) + Reinic, et al., arXiv:2507.21236

💡 Insert dis-entanglers to better cope with boundary-law at reasonable cost



TF-Ising close to critical

Heisenberg triangular



- * Overview of TN approaches for 2D
- * Tree Tensor Networks (TTN):
 - * Basic definitions & operations
 - * App 1: statics of (bosonic) fractional quantum Hall
 - * App 2: dynamics of interfaces (2D Ising)
 - * other applications & extensions
- * Projected Entangled Pair States (PEPS):
 - * Variational algorithm for infinite systems
 - * App: statics of (bosonic) fractional quantum Hall
 - * Stochastic reconfiguration for finite systems
- * Conclusion

PROJECTED ENTANGLED PAIR STATES (PEPS)

VARIATIONAL CALCULATIONS FOR INFINITE SYSTEMS

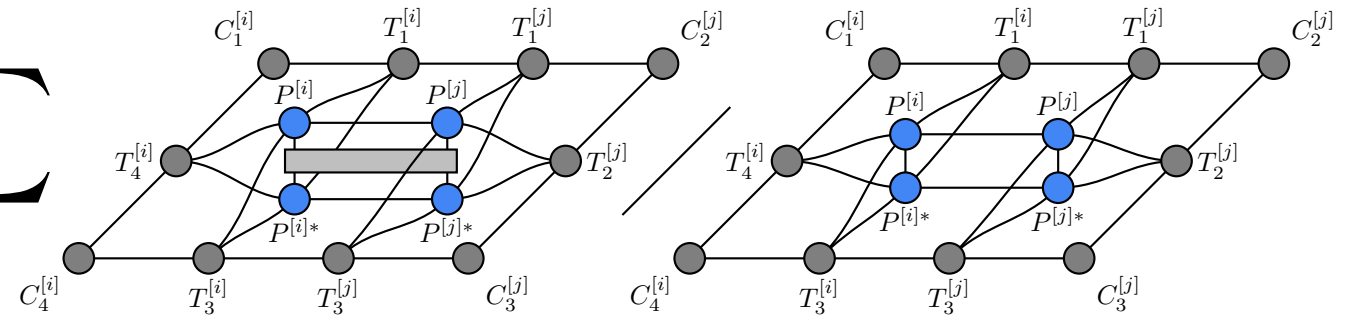
J. Naumann et al. (MR), SciPost Phys. Lect. Notes 86 ('24)

J. Naumann et al. (MR), Phys. Rev. B 111, 235116 ('25)

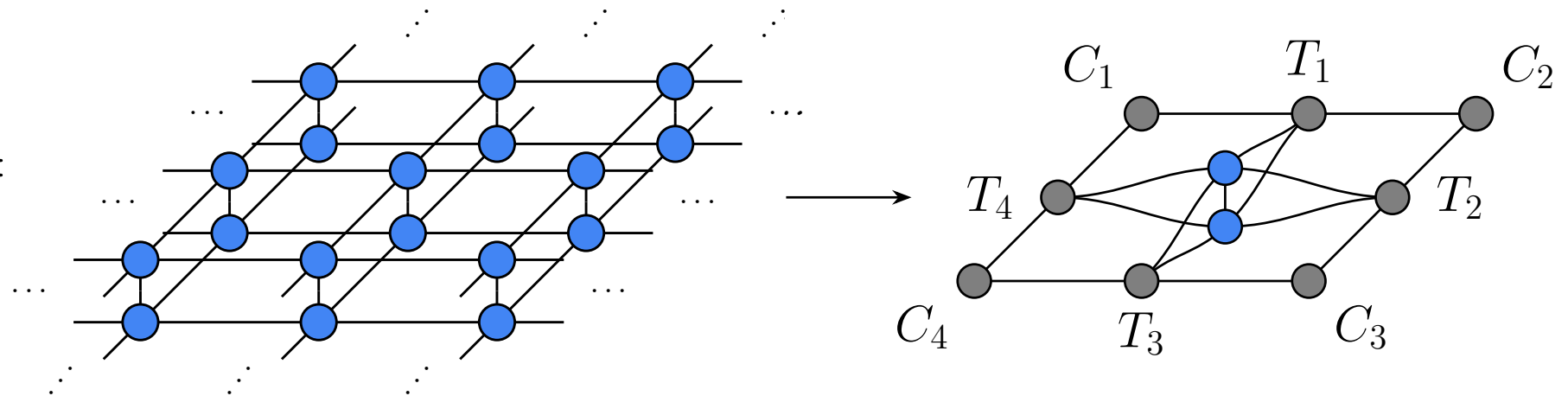




$$\operatorname{argmin}_{|\psi\rangle \in \text{PEPS}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \sum$$



$$\langle \psi | \psi \rangle =$$

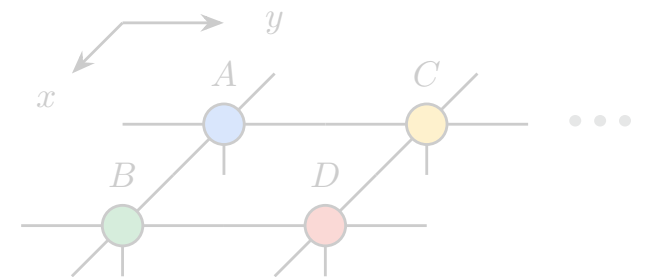
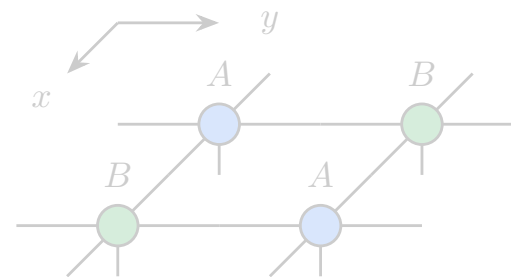
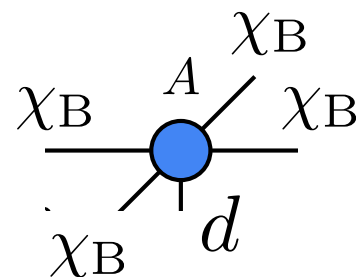


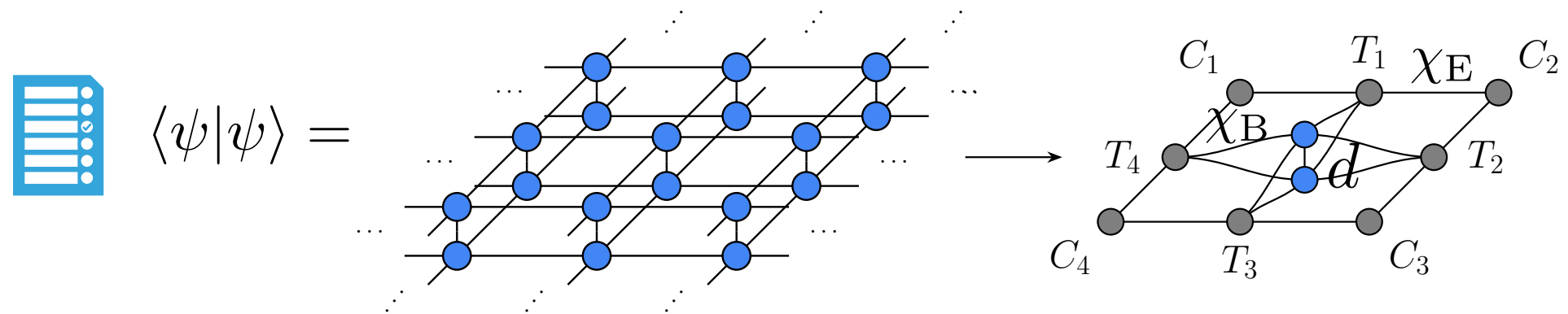
Contraction of 2D loopy networks scale exponentially with size



Attack systems directly in the thermodynamic limit!

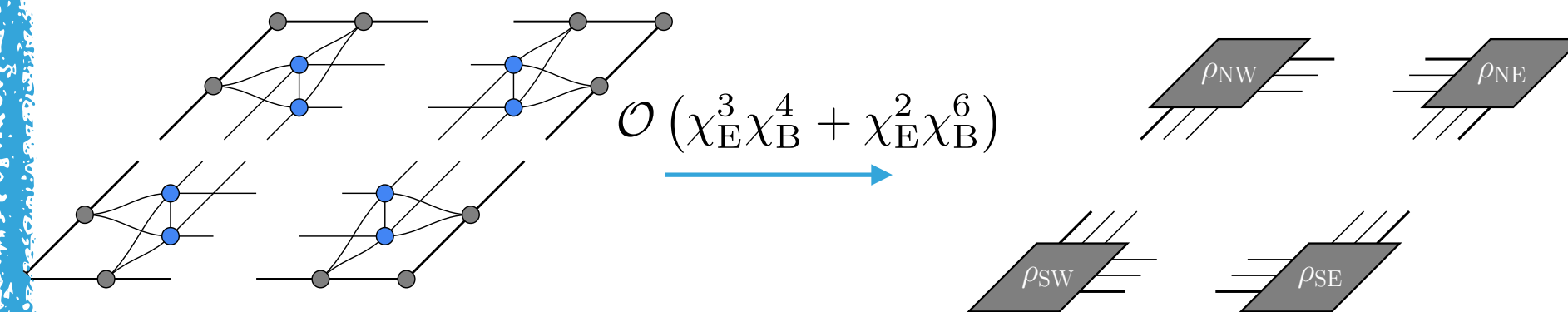
Different unit cells:

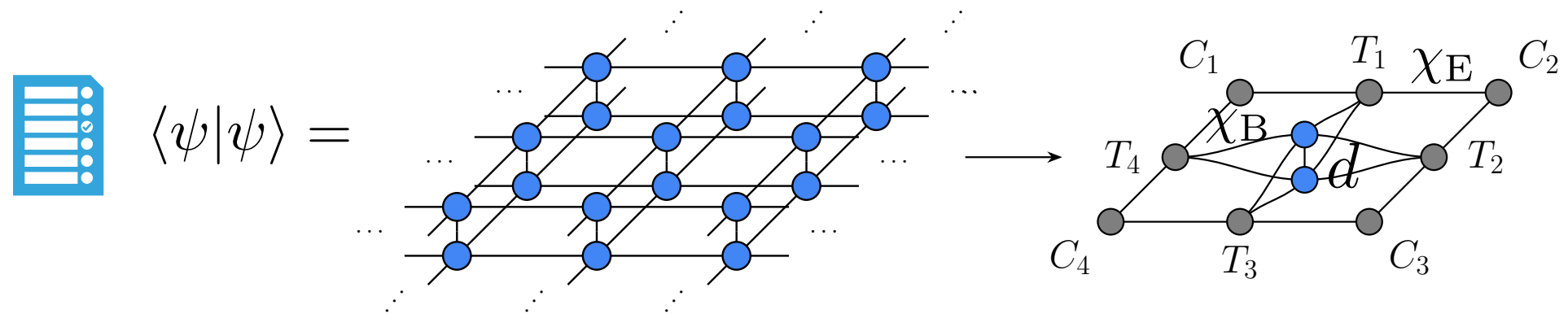




💡 Attack systems directly in the thermodynamic limit!

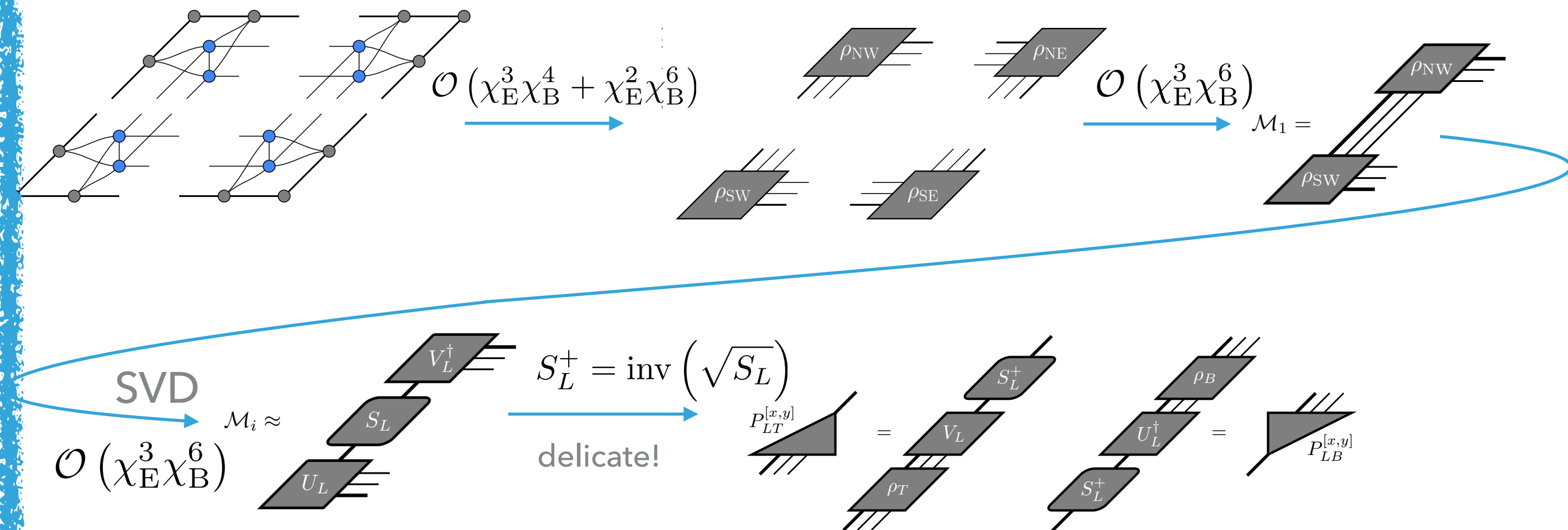
▽ Computationally intensive tensor operations





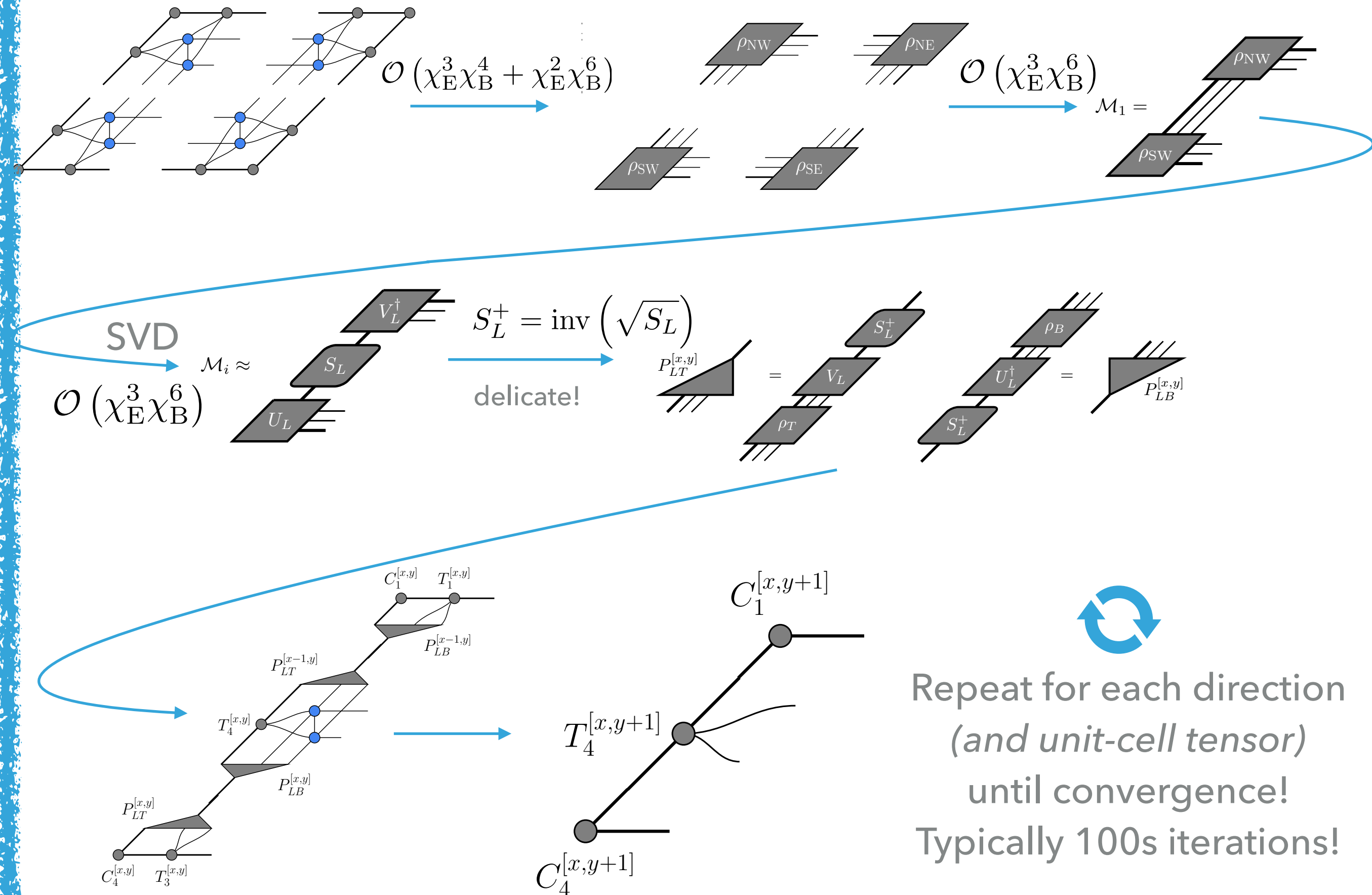
💡 **Attack systems directly in the thermodynamic limit!**

▽ Computationally intensive tensor operations



CORNER TRANSFER-MATRIX RG

36



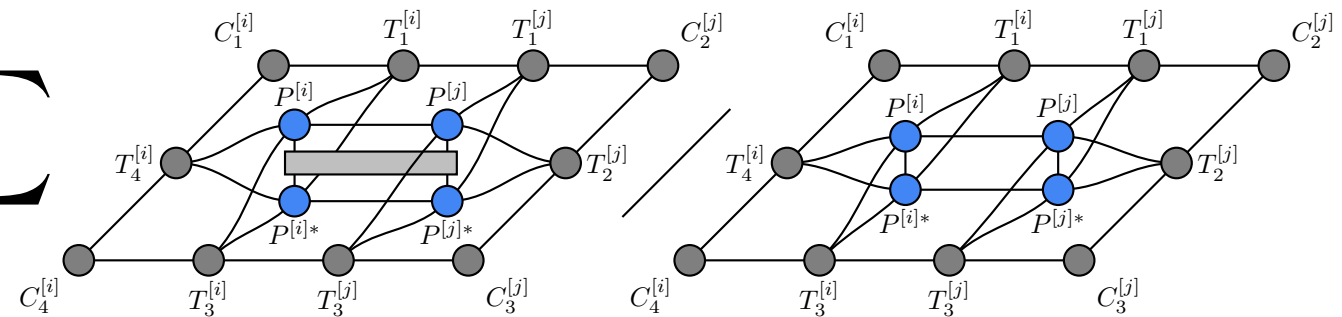
VARIATIONAL I-PEPS



1st pioneering work by P. Corboz PRB **94**, 035133 (2016) ³⁷



$$\operatorname{argmin}_{|\psi\rangle \in \text{PEPS}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \sum$$



$$\langle \Psi | \Psi \rangle =$$

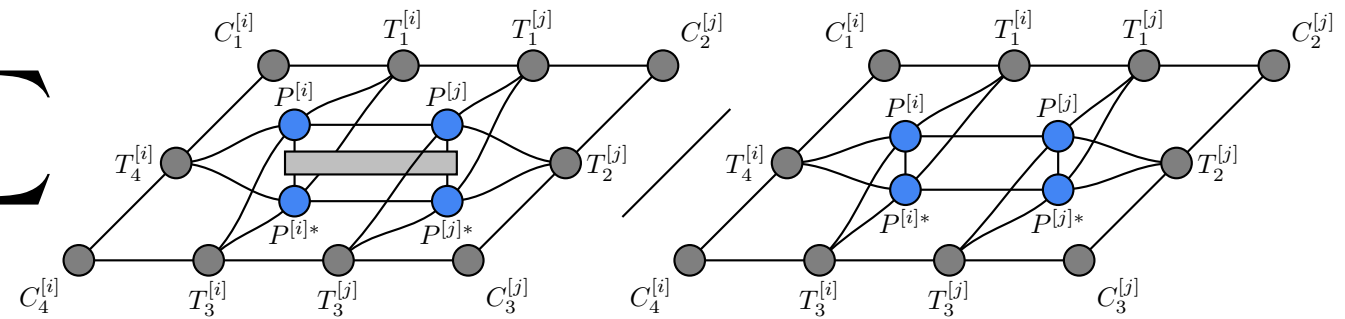
VARIATIONAL I-PEPS



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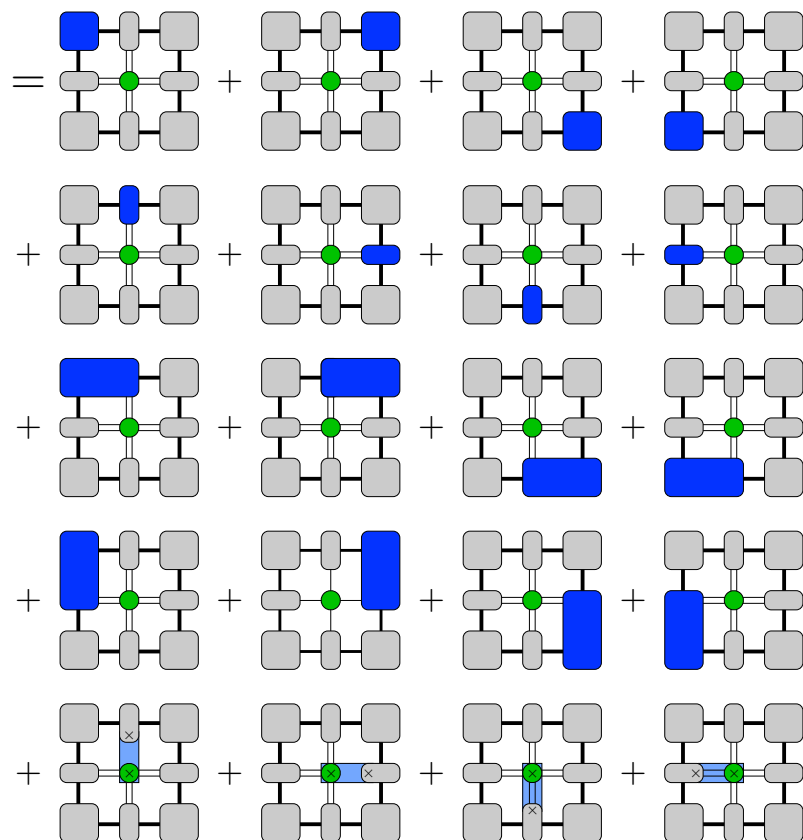


$$\operatorname{argmin}_{|\psi\rangle \in \text{PEPS}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \sum$$



$$\langle \Psi | \Psi \rangle =$$

$$\langle \Psi | \hat{H} | \Psi \rangle =$$



$$\begin{aligned} \tilde{C}_1 &= \text{blue square} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots \\ \tilde{T}_4 &= \text{blue square} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \\ \tilde{C}_{v1} &= \text{blue square} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \end{aligned}$$

$$\begin{aligned} \tilde{C}'_1 &= \text{blue square} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} \\ \tilde{C}'_{h1} &= \text{blue square} = \text{diagram 1} \\ \tilde{C}'_{v1} &= \text{blue square} = \text{diagram 1} + \text{diagram 2} \\ T\tilde{T}'_4 &= \text{blue square} = \text{diagram 1} + \text{diagram 2} \\ \tilde{T}'_4 &= \text{blue square} = \text{diagram 1} + \text{diagram 2} \\ T'^o_4 &= \text{blue square} = \text{diagram 1} + \text{diagram 2} \\ \tilde{C}'_{v4} &= \text{blue square} = \text{diagram 1} + \text{diagram 2} \\ \tilde{C}'_{h4} &= \text{blue square} = \text{diagram 1} \\ \tilde{C}'_4 &= \text{blue square} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} \end{aligned}$$

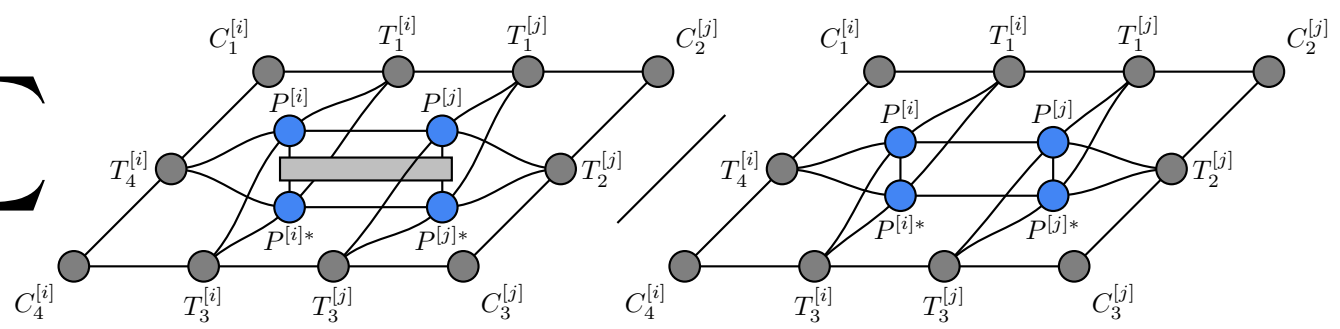
VARIATIONAL I-PEPS



1st pioneering work by P. Corboz PRB **94**, 035133 (2016) ³⁷



$$\operatorname{argmin}_{|\psi\rangle \in \text{PEPS}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \sum$$



$$\langle \Psi | \Psi \rangle = \sum_{x,y} \text{[Tensor Network Diagram]}$$



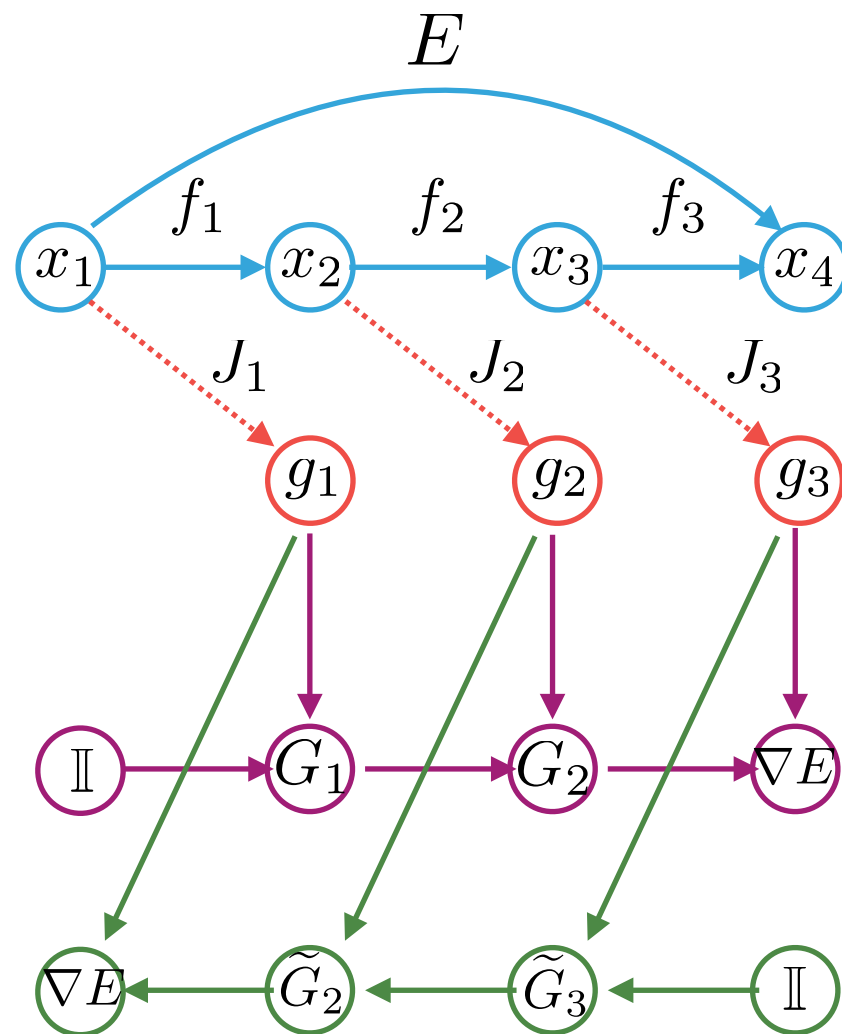
Effective generalized eigenvalue problem

$$\frac{\partial \langle \psi | H | \psi \rangle}{\partial \text{[Parameter]}} \rightarrow \text{[Diagram]} = E \text{[Diagram]} \leftarrow \frac{\partial \langle \psi | \psi \rangle}{\partial \text{[Parameter]}}$$

$$\langle \Psi | \hat{H} | \Psi \rangle = \text{[Sum of 20 tensor network diagrams]}$$

$$\begin{aligned} \tilde{C}_1 &= \text{[Diagram]} = \text{[Sum of diagrams]} + \dots \\ \tilde{T}_4 &= \text{[Diagram]} = \text{[Sum of diagrams]} + \dots \\ \tilde{C}_{v1} &= \text{[Diagram]} = \text{[Sum of diagrams]} + \dots \end{aligned}$$

$$\begin{aligned} \tilde{C}'_1 &= \text{[Diagram]} = \text{[Sum of diagrams]} \\ \tilde{C}'_{h1} &= \text{[Diagram]} = \text{[Sum of diagrams]} \\ \tilde{C}'_{v1} &= \text{[Diagram]} = \text{[Sum of diagrams]} \\ \tilde{T}'_4 &= \text{[Diagram]} = \text{[Sum of diagrams]} \\ \tilde{C}'_{v4} &= \text{[Diagram]} = \text{[Sum of diagrams]} \\ \tilde{C}'_{h4} &= \text{[Diagram]} = \text{[Sum of diagrams]} \\ \tilde{C}'_4 &= \text{[Diagram]} = \text{[Sum of diagrams]} \end{aligned}$$



$x_i \in \mathbb{R}^{n_i}$ = input, intermediate results, outputs

f_j = "elementary" functions with known Jacobians

$$J_j = \frac{\partial f_j}{\partial x_j} : \mathbb{R}^{n_j} \rightarrow \mathbb{R}^{n_{j+1}} \times \mathbb{R}^{n_j}$$

Chain-rule: $\nabla E \equiv \frac{\partial E}{\partial x_1} = g_3 \cdot g_2 \cdot g_1$

Forward-mode $j \uparrow$

$$G_j := g_j \cdot G_{j-1} \quad G_0 := \mathbb{I}_{n_1} \quad \nabla E = G_3$$

Backward-mode $j \downarrow$

$$\tilde{G}_j := \tilde{G}_{j+1} \cdot g_j \quad \tilde{G}_4 = \mathbb{I}_{n_4} \quad \nabla E = \tilde{G}_1$$

Backward-mode way more convenient for iPEPS:

i) cheaper operations: $n_{\text{out}} = 1 \ll n_{\text{in}} = d \chi_B^4$



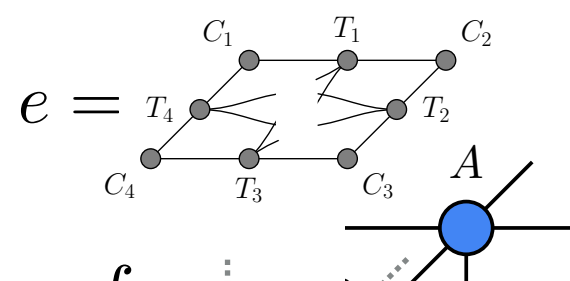
ii) fix-point condition allows to keep memory light! $e^*(A) = f(A, e^*(A))$

VARIATIONAL I-PEPS: AUTOMATIC DIFFERENTIATION 39

▽ Backward-mode way more convenient for iPEPS:

i) cheaper operations: $n_{\text{out}} = 1 \ll n_{\text{in}} = d \chi_B^4$

ii) fix-point condition allows to keep memory light! $e^*(A) = f(A, e^*(A))$



$$\partial_A e^* = \sum_{n=0}^{\infty} (\partial_e f)^n \cdot \partial_A f$$

$$\partial_A e^* = \partial_A f + \partial_e f \cdot \partial_A e^*$$

$$(\partial_A f + \partial_e f \cdot \partial_A e_3)$$

$$(\partial_A f + \partial_e f \cdot \partial_A e_2)$$

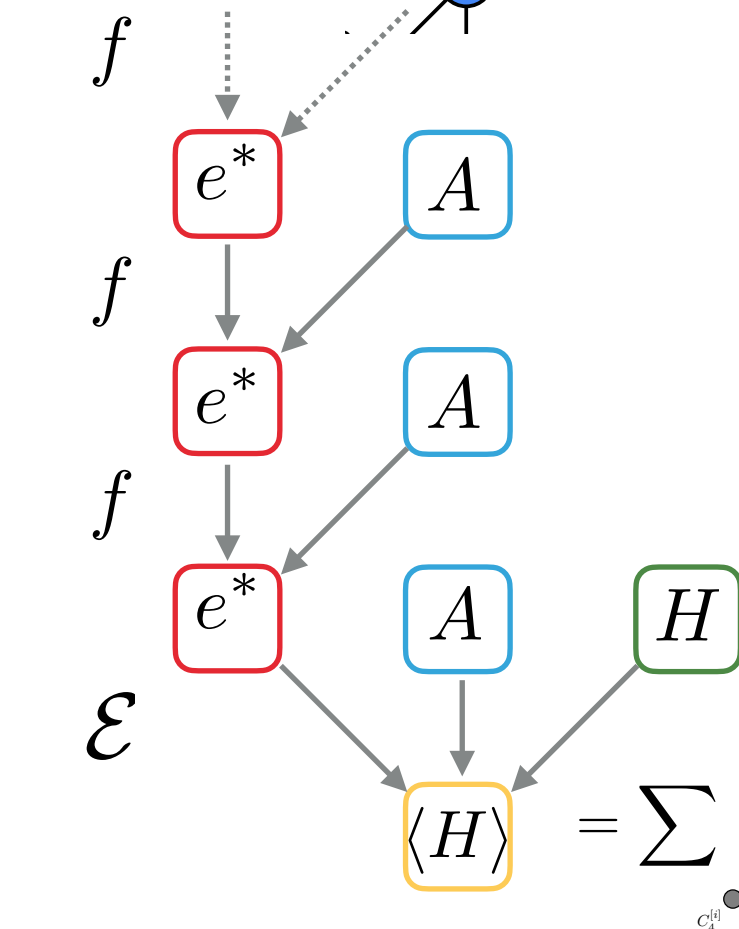
$$\partial_A e_0 = \partial_A f + \partial_e f \cdot \partial_A e_1$$

$$(1 - \partial_e f) \cdot \partial_A e^* = \partial_A f$$

linear problem, efficient!

$$\partial_A \langle H \rangle = \partial_A \mathcal{E} + \partial_{e^*} \mathcal{E} \cdot \partial_A e^*$$

...now use your favourite gradient search method...

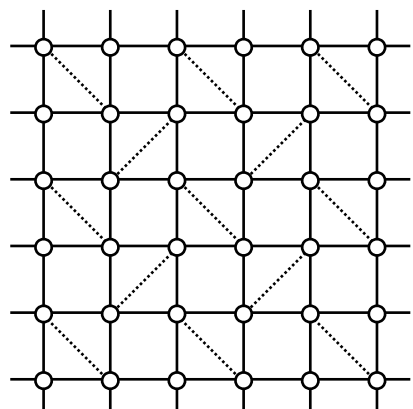


BUT watch out that gauge degrees of freedom do not trick you!

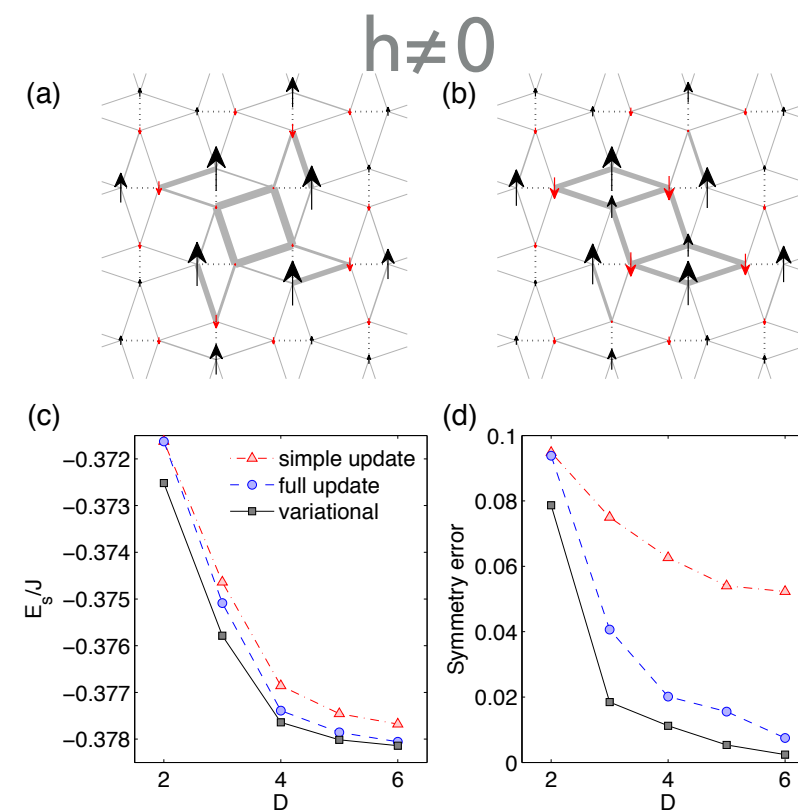
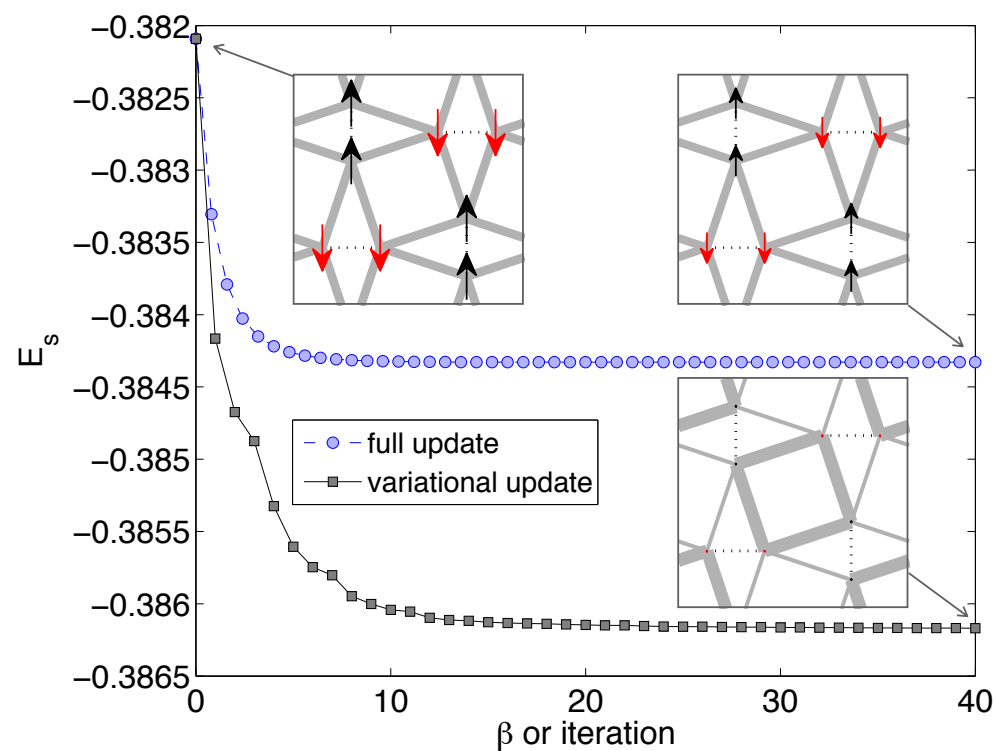
Tang et al.,
arXiv:2508.10822

BENCHMARKS: $S=1/2$ MODELS

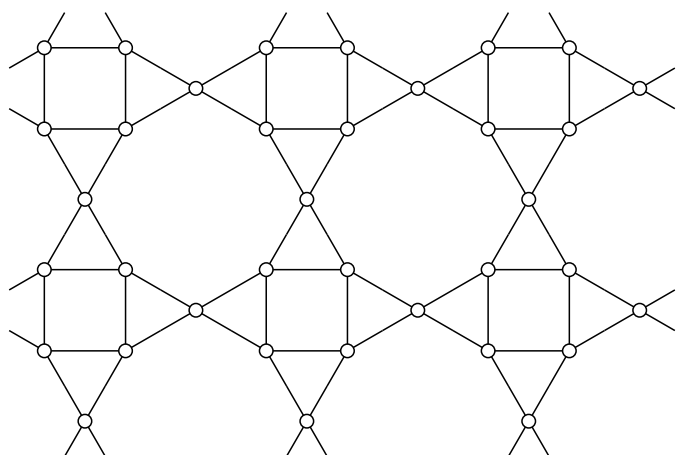
Shastry-Sutherland



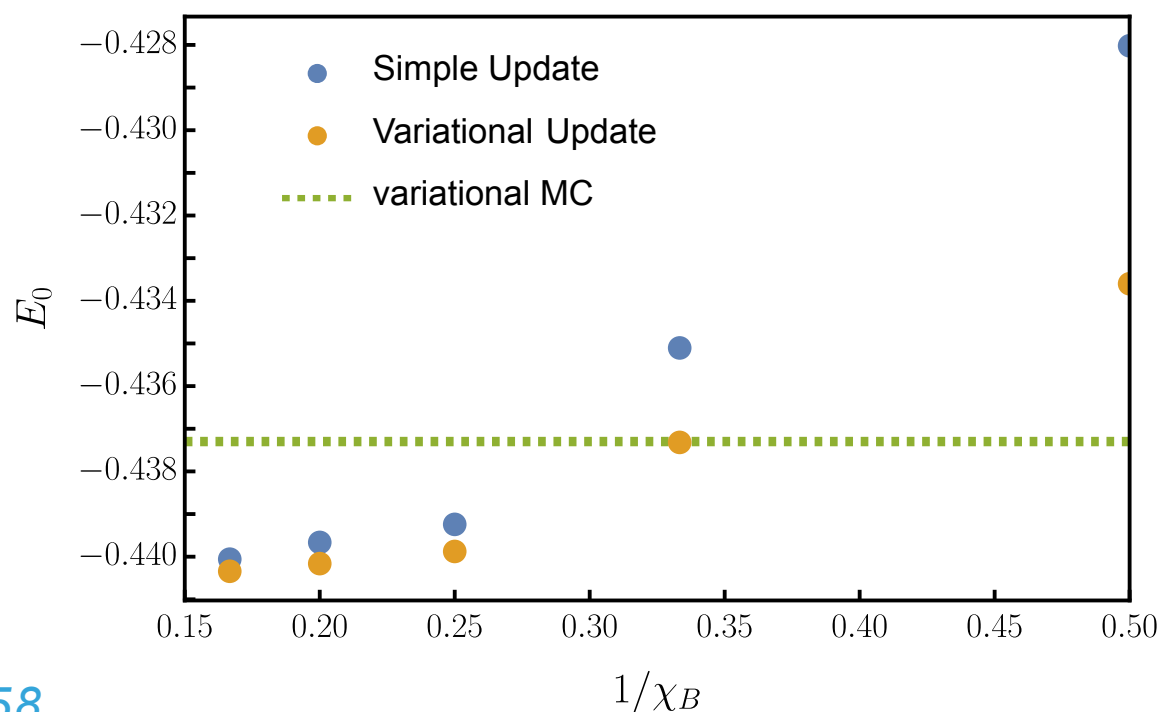
P. Corboz PRB 94, 035133 (2016)



Heisenberg



E. Weerda, J. Naumann, **MR**,
J. Eisert, P. Schmoll, [arXiv:2308.12358](https://arxiv.org/abs/2308.12358)



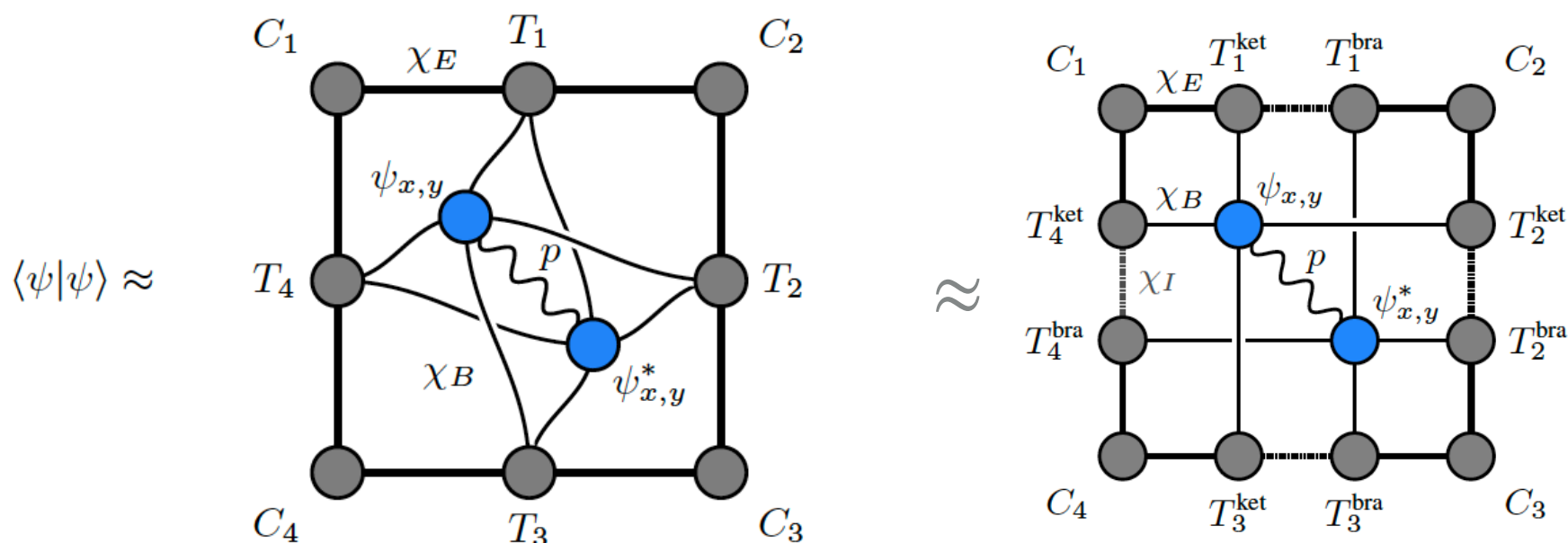
χ_B	E_0 (SU)	E_0 (VU)
2	-0.42802	-0.43360
3	-0.43511	-0.43732
4	-0.43924	-0.43988
5	-0.43967	-0.44016
6	-0.44006	-0.44034

ACCELERATING I-PEPS: SPLIT-CTMRG

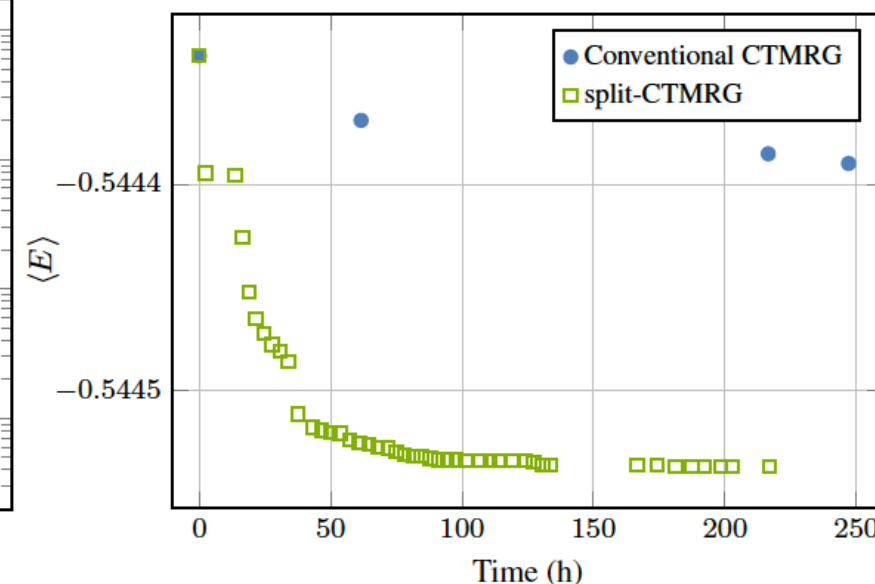
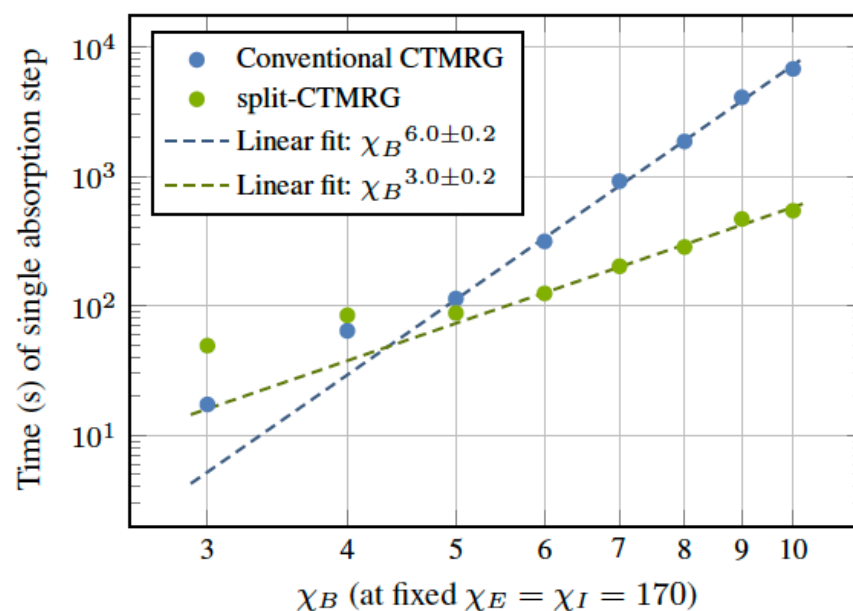
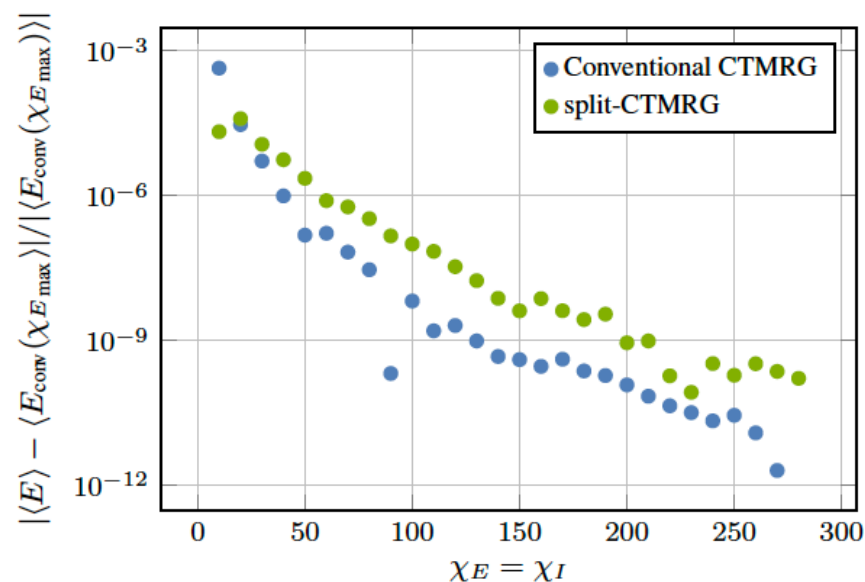
41

J. Naumann et al. (MR), Phys. Rev. B 111, 235116 ('25)

💡 Split 4-leg tensors into pairs of 3-leg ones...



... get comparable precision much cheaper / higher precision much faster



PROJECTED ENTANGLED PAIR STATES (PEPS)

VARIPEPS FOR (BOSONIC) FRACTIONAL HALL STATES

E.L. Weerda, **MR**, PRB 109, L241117 (2024)

HARPER-HOFSTADTER MODEL

43

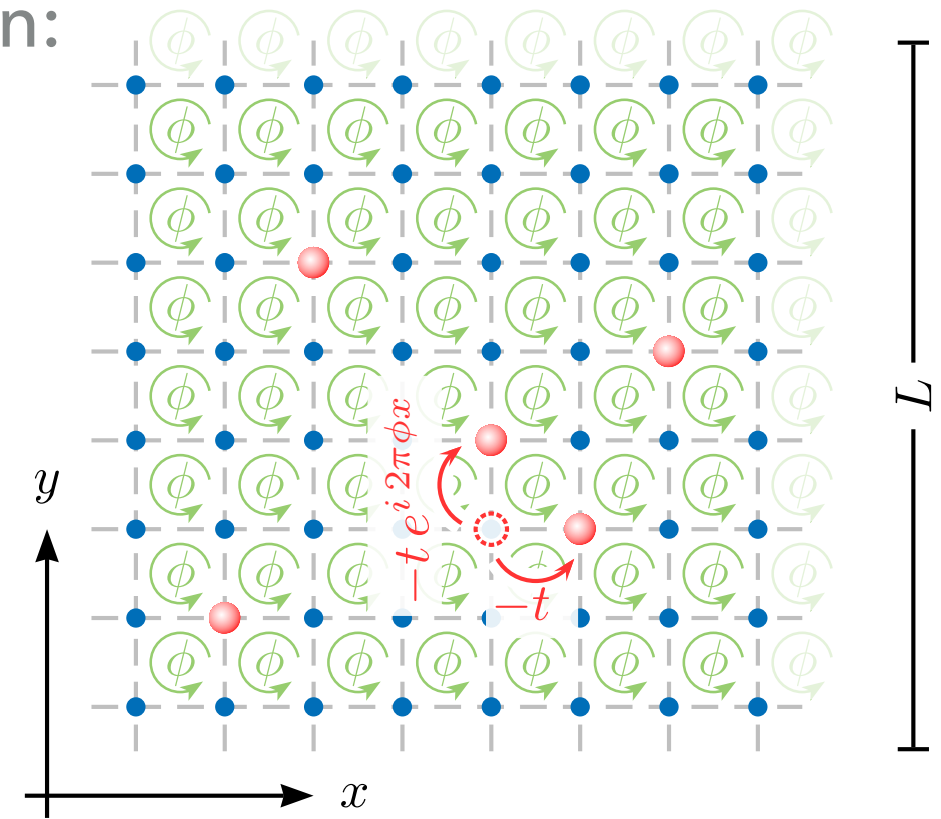
simplest **lattice discretization** of QH Hamiltonian:
Bose-Hubbard model + piercing magnetic flux

*three length-scales
in competition
 $\nu = \rho/\alpha$*

$$\alpha = \phi/\Phi_0$$

$$a/\ell_B = \sqrt{2\pi\alpha}$$

$$\rho = N/L^2$$

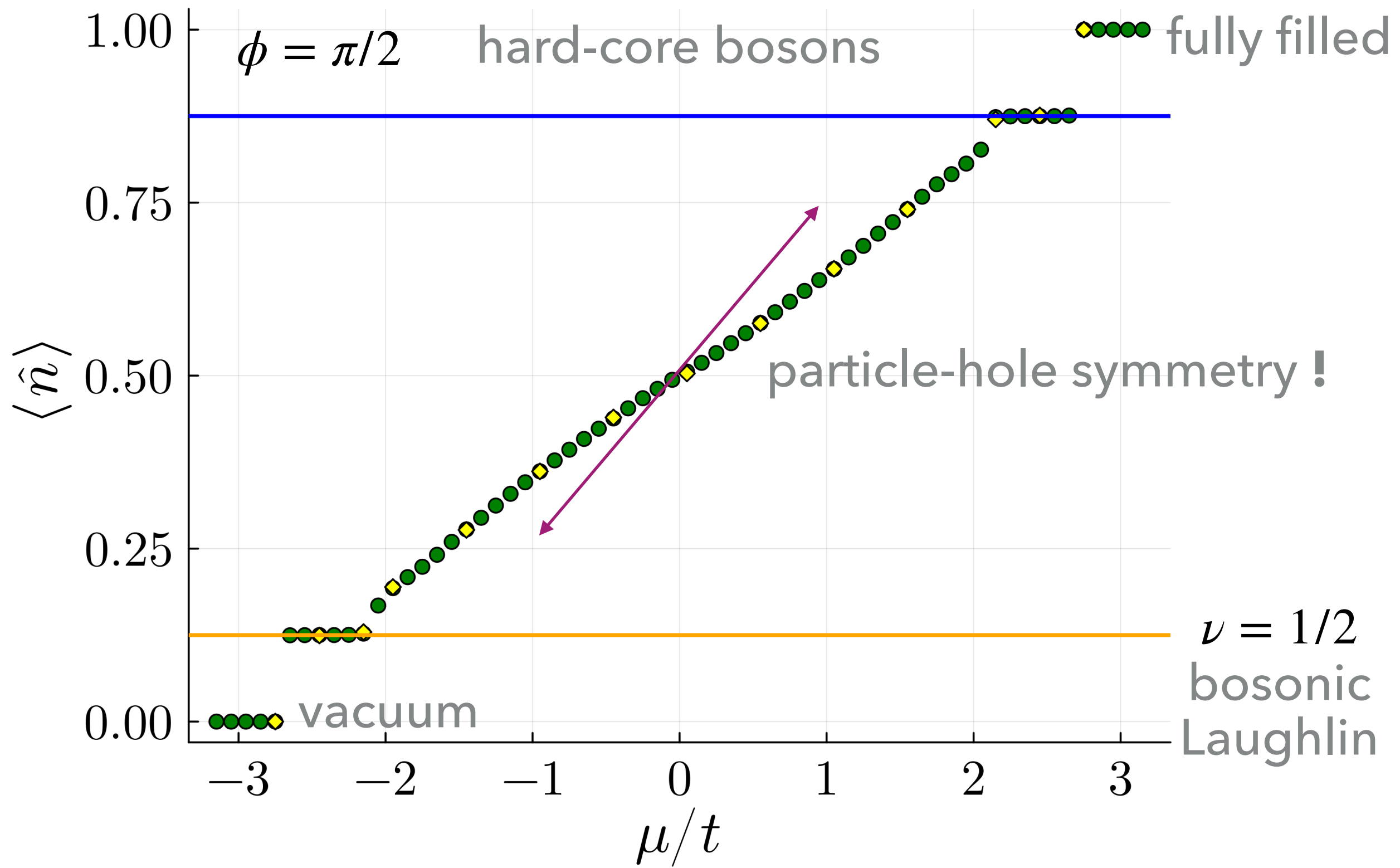


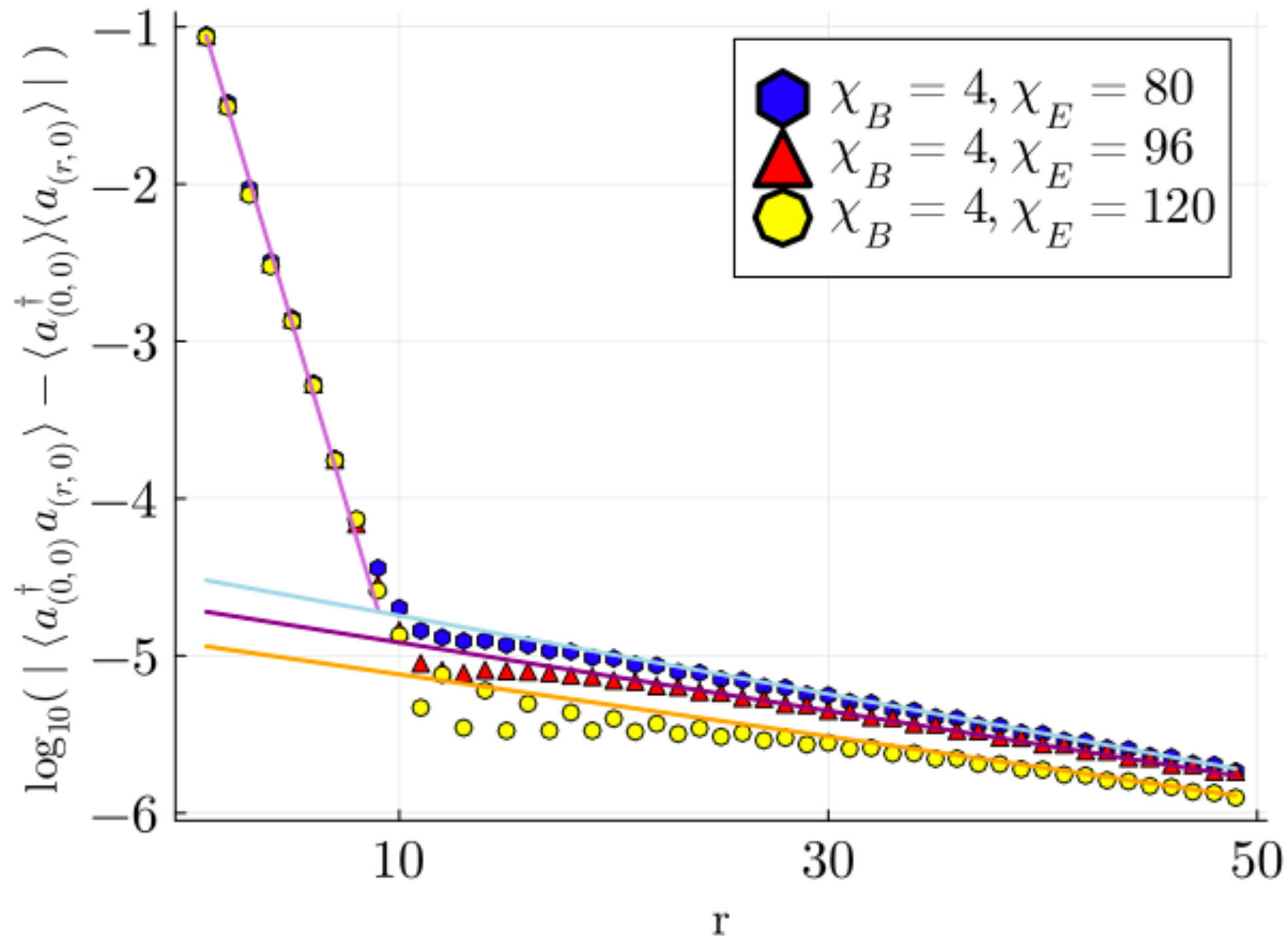
same model as before for the TTN :-)

FQH-IPEPS: PLATEAUS

44

Harper-Hofstadter model with tunable density:
expect plateaus at commensurate $\nu = \langle \hat{n} \rangle 2\pi/\phi$





gapped bulk

$$\xi/\ell_B \simeq 1.32$$

“fake” long-range tails

Hasik, et al., PRL 129, 177201 (2022)

FTN: Gerster, et al. (MR), PRB 96, 195123 (2017)

physical edge modes $\Leftrightarrow$ lowest entanglement spectrum

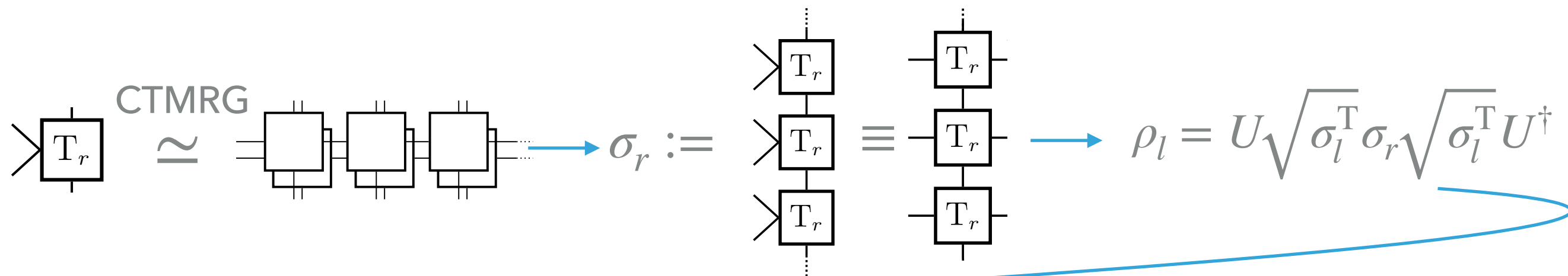
Li & Haldane, PRL 101, 010504 (2008)

$$\rho_l = \text{Tr}_r(|\psi\rangle\langle\psi|) = e^{-H_{\text{ent}}}$$

💡 use bulk-boundary correspondence for (i)PEPS

Cirac, Poilblanc, Schuch, Verstraete, PRB 83, 245134 (2011)

Haegeman & Verstraete, Ann. Rev. Cond. Mat. Phys. 8, 355 (2017)



$$\text{Sp}(\rho_l) = \text{Sp}(\sigma_l \sigma_r) = \text{Sp}(\begin{matrix} T_r & T_l \\ T_r & T_l \\ T_r & T_l \end{matrix})$$

MPO with bond χ_E^2

VUMPS techniques for ground & excited at given momentum

Haegeman, et al., PRL 111, 080401 (2013)

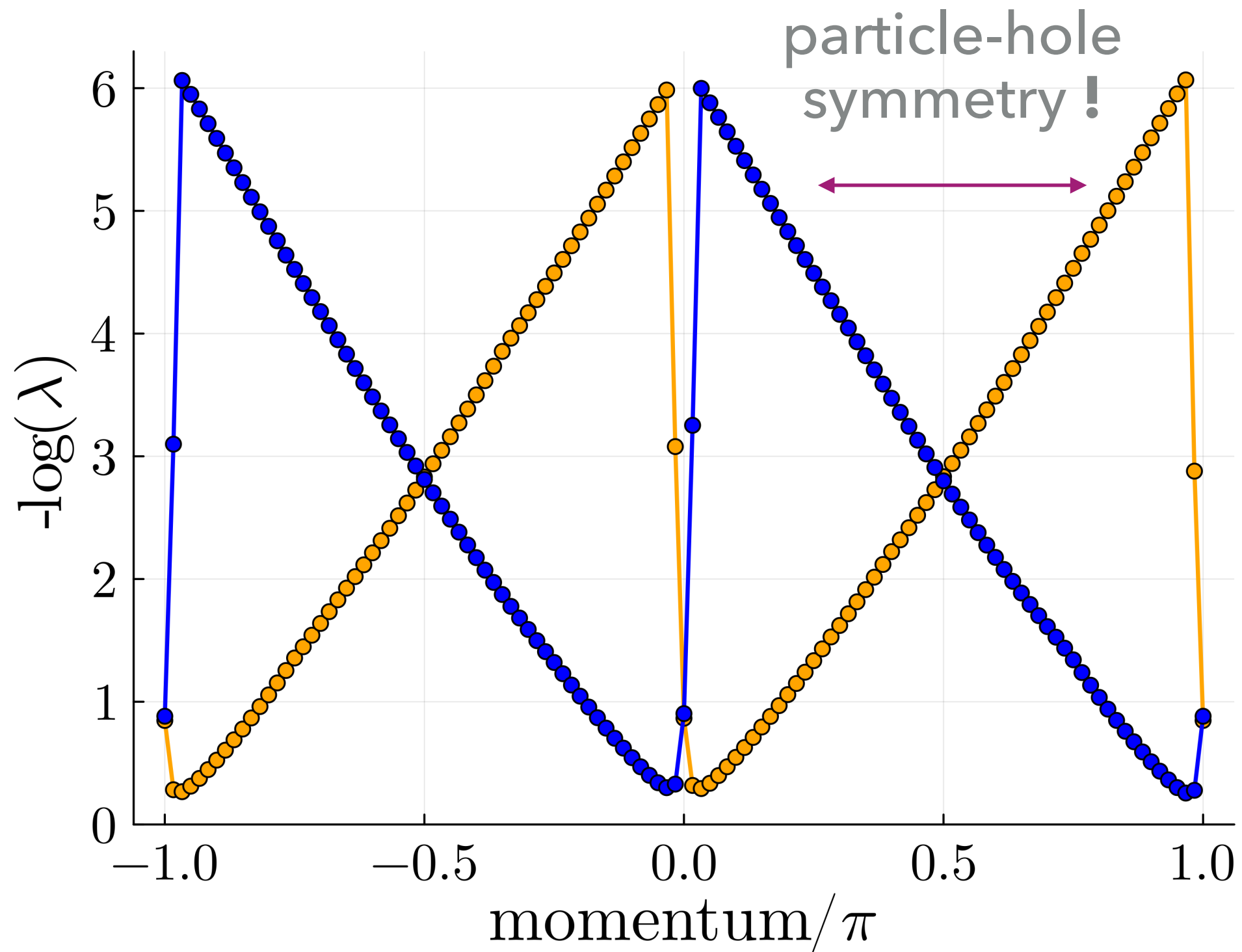
Vanderstraeten, et al., SciPost Phys. Lect. Notes, 7 (2019)

FQH-IPEPS: EDGE MODES, CHIRAL!

47

physical edge modes $\Leftrightarrow$ lowest entanglement spectrum

Li & Haldane, PRL 101, 010504 (2008)



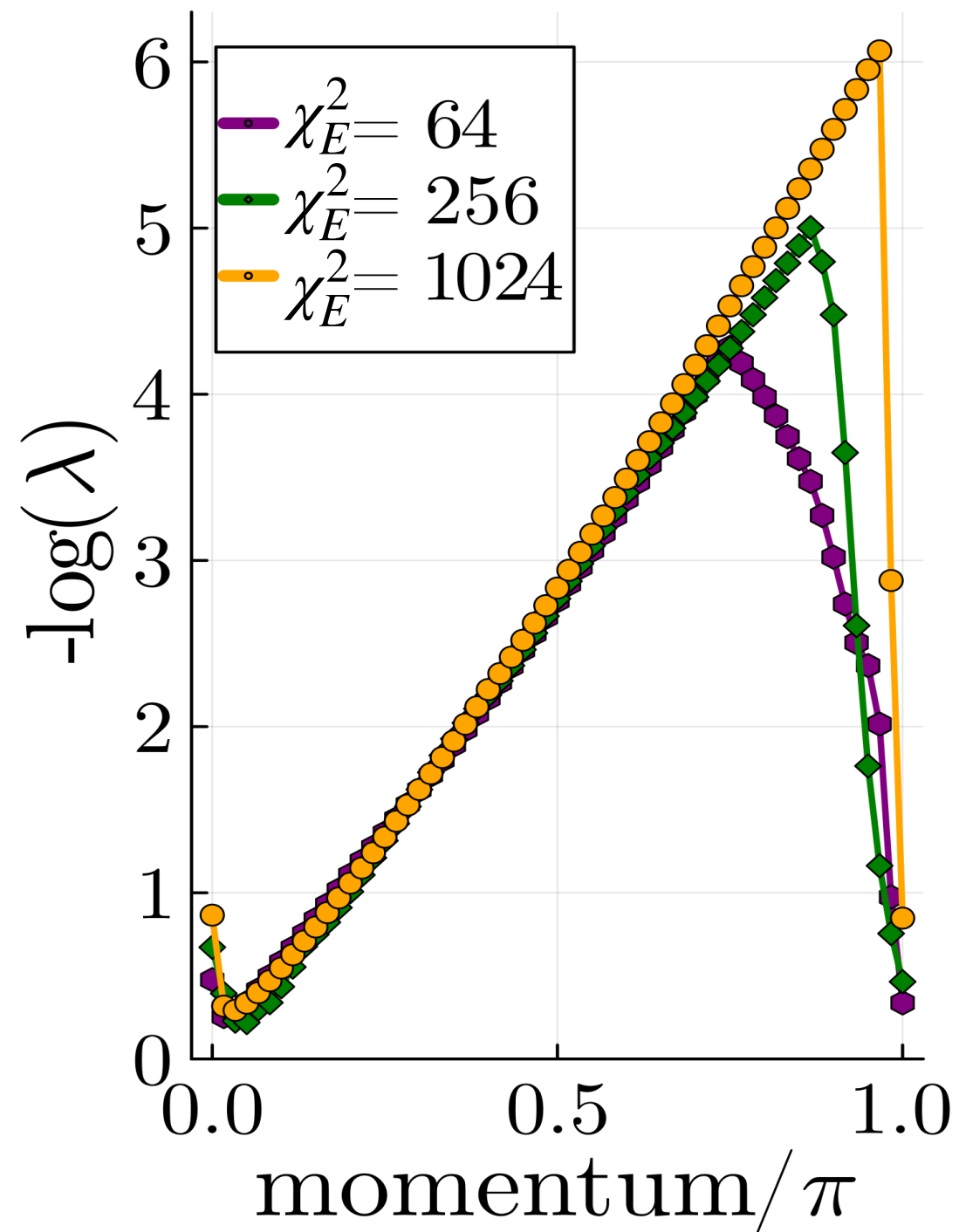
FQH-IPEPS: EDGE MODES, CHIRAL!

47

physical edge modes $\iff$ lowest entanglement spectrum

Li & Haldane, PRL 101, 010504 (2008)

sharper chirality
with more refined
approximation of
iPEPS contraction



FQH-IPEPS: EDGE MODE COUNTING

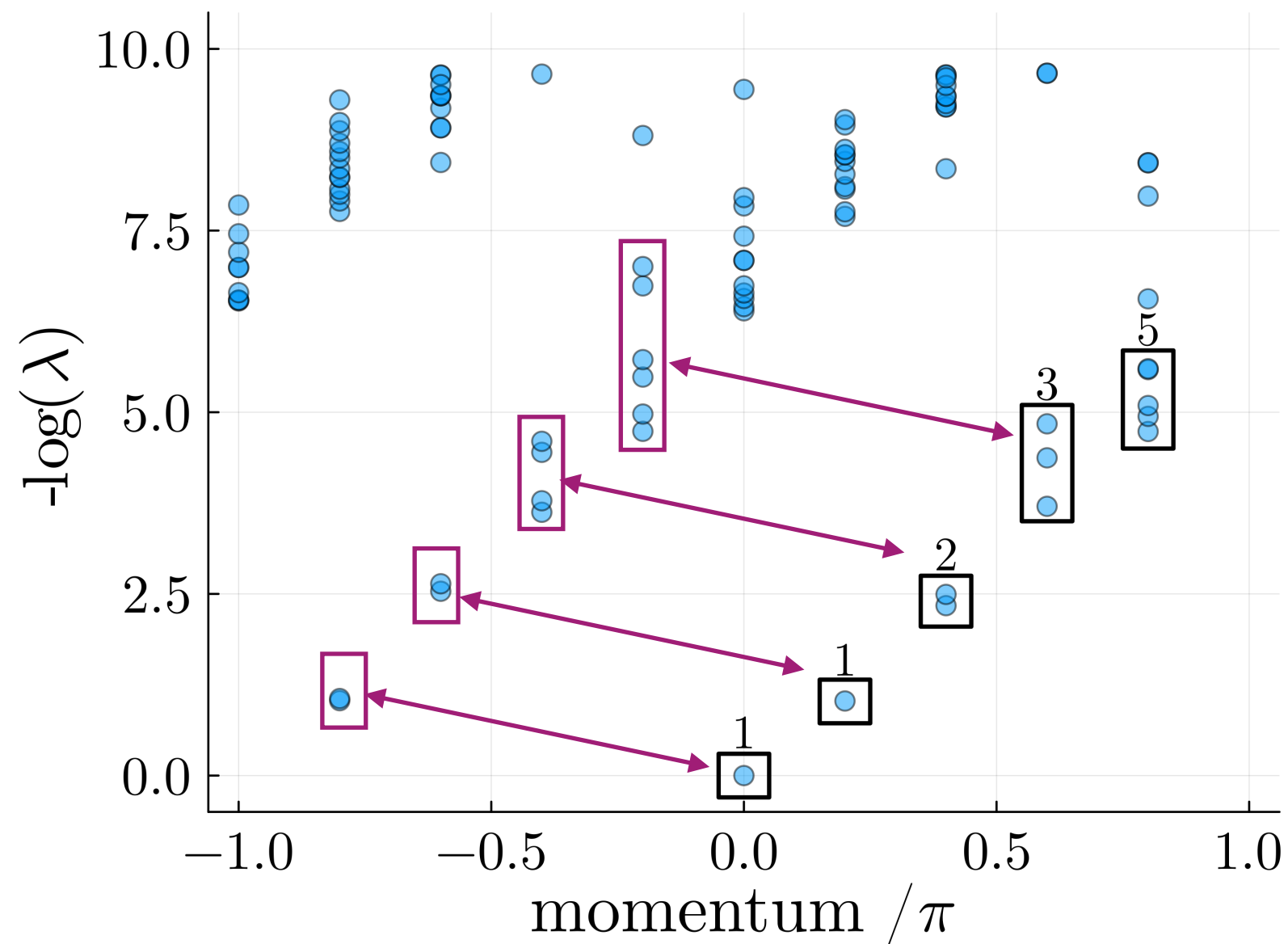
48

physical edge modes $\Leftrightarrow$ lowest entanglement spectrum

Li & Haldane, PRL 101, 010504 (2008)

finite cylinder $\Rightarrow$ discrete momenta $\Rightarrow$ degeneracy counting

for Laughlin expect chiral boson CFT counting, i.e., # integer partitions: (1,1,2,3,5,7,11,15,...)



second branch at $k < 0$ with doubled counting (extra particle/hole)

Cincio & Vidal, PRL 110, 067208 (2013)

PROJECTED ENTANGLED PAIR STATES (PEPS)

FINITE SYSTEMS VIA STOCHASTIC RECONFIGURATION

[Alcalde Puente, et al. \(MR\), PRB 111, 195120 \('25\)](#)



Imaginary-**TDVP** (a.k.a. **natural gradient**) for N_P param. θ (tensor entries)

$$|\dot{\Psi}(\theta)\rangle = -H |\Psi(\theta)\rangle \longrightarrow \sum_i \underbrace{\frac{\partial \langle S | \Psi(\theta) \rangle}{\partial \theta_i}}_{\tilde{O}_{S,i}} \dot{\theta}_i = - \underbrace{\langle S | H | \Psi(\theta) \rangle}_{\tilde{E}_S^{\text{loc}}}, \quad \forall S \in \mathcal{H}$$

not (exactly) solvable, since $N_P \ll \dim(\mathcal{H})$



Best approx. by N_S relevant **samples** in $\mathcal{H}$

$$\min_{\dot{\theta}} \sum_{\{S\}} \frac{|\langle \Psi | S \rangle|^2}{\langle \Psi | \Psi \rangle} \left\| \sum_i \underbrace{\frac{\tilde{O}_{S,i}}{\langle \Psi | S \rangle}}_{O_{S,i}} \dot{\theta}_i - \underbrace{\frac{\tilde{E}_S^{\text{loc}}}{\langle \Psi | S \rangle}}_{E_S^{\text{loc}}} \right\|^2$$

$\frac{1}{N_S} \sum_{S \sim p_\Psi} \rho_\Psi(S)$



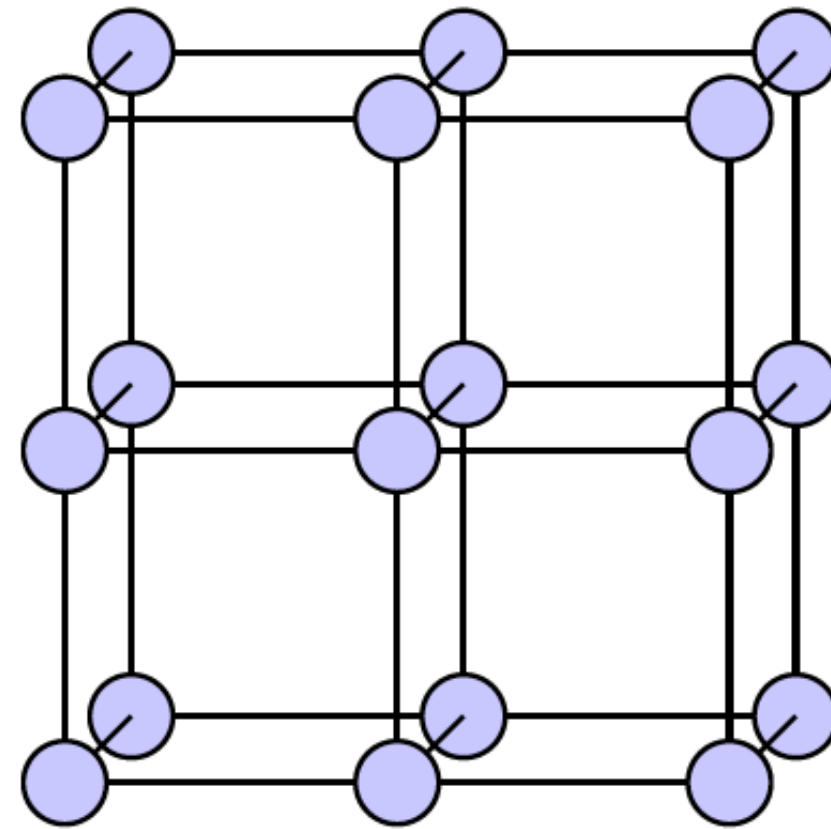
smart pseudo-inversion by **minSR** [$N_S \ll N_P$]
(minimum-step Stochastic Reconfiguration)

$$\dot{\theta} = O^\dagger \underbrace{(OO^\dagger)^{-1}}_T E_{\text{loc}}$$

Chen, Heyl,
Nat. Ph. 20, 1476 ('24)

FINITE-PEPS: SAMPLING, ISSUES & FEATURES

Vieijra, et al., PRB 104, 235141 ('21)

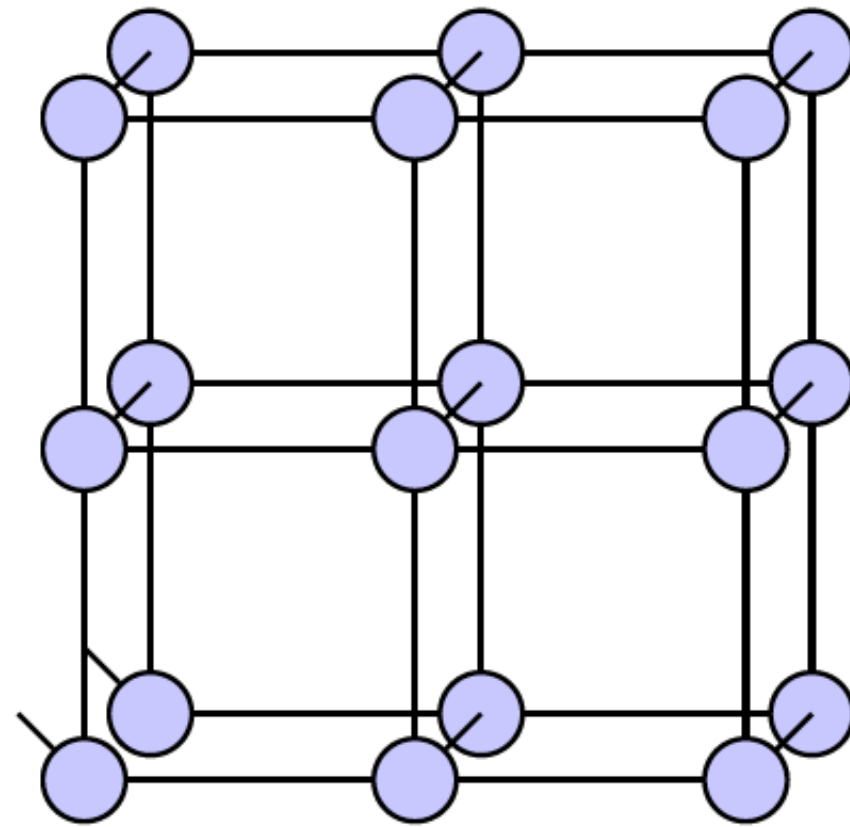


$$\langle \Psi | \Psi \rangle$$

FINITE-PEPS: SAMPLING, ISSUES & FEATURES

Vieijra, et al., PRB 104, 235141 ('21)

A Compute $\rho(S_1)$

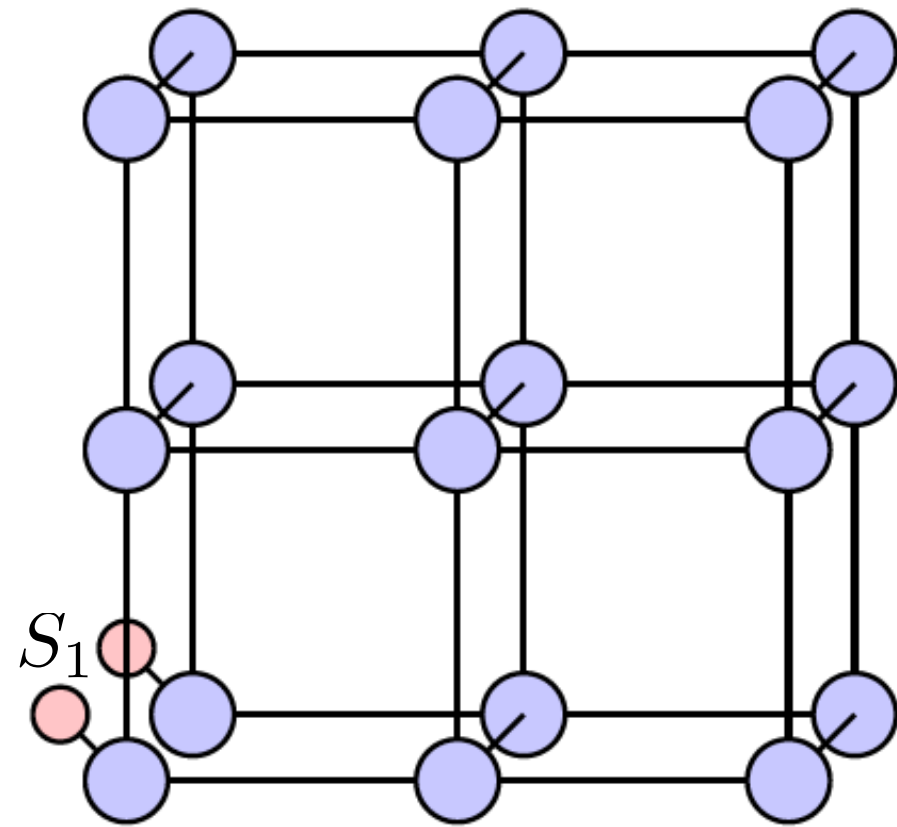


$\rho(S_1)$

FINITE-PEPS: SAMPLING, ISSUES & FEATURES

Vieijra, et al., PRB 104, 235141 ('21)

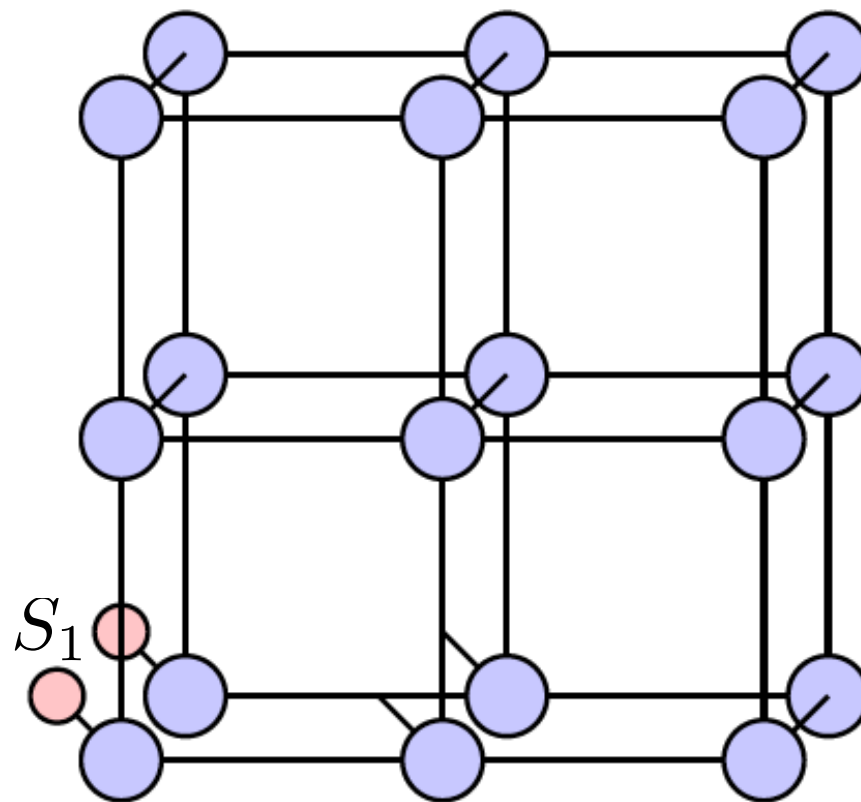
- A Compute $\rho(S_1)$
- B Sample S_1 accordingly



FINITE-PEPS: SAMPLING, ISSUES & FEATURES

Vieijra, et al., PRB 104, 235141 ('21)

- A** Compute $\rho(S_1)$
- B** Sample S_1 accordingly
- C** Compute $\rho(S_2 | S_1)$

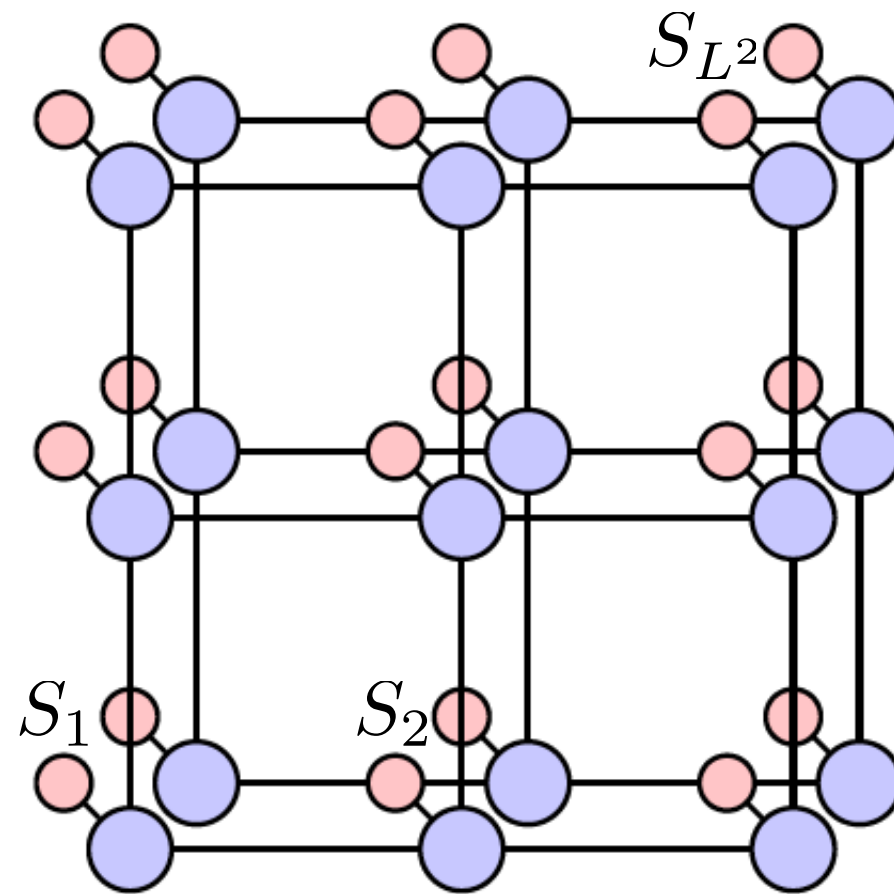


FINITE-PEPS: SAMPLING, ISSUES & FEATURES

Vieijra, et al., PRB 104, 235141 ('21)

- A** Compute $\rho(S_1)$
- B** Sample S_1 accordingly
- C** Compute $\rho(S_2 | S_1)$
- D** Repeat until S_{L^2}

***No autocorrelation
& parallelizable !!!***

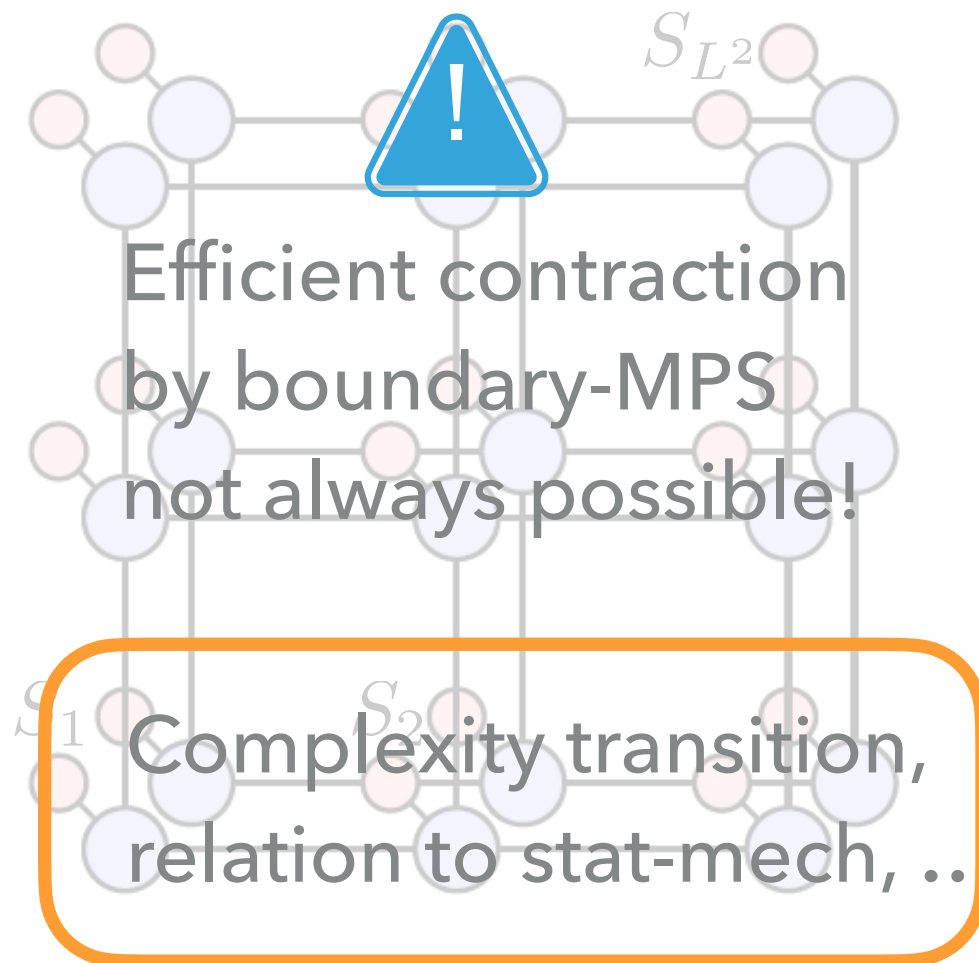


FINITE-PEPS: SAMPLING, ISSUES & FEATURES

Vieijra, et al., PRB 104, 235141 ('21)

- A Compute $\rho(S_1)$
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Vasseur, ..., PRB 100, 134203 ('19)

Chen, ..., PRX-Q 6, 010312 ('25)

McGinley, ..., PRX 15, 021059 ('25)

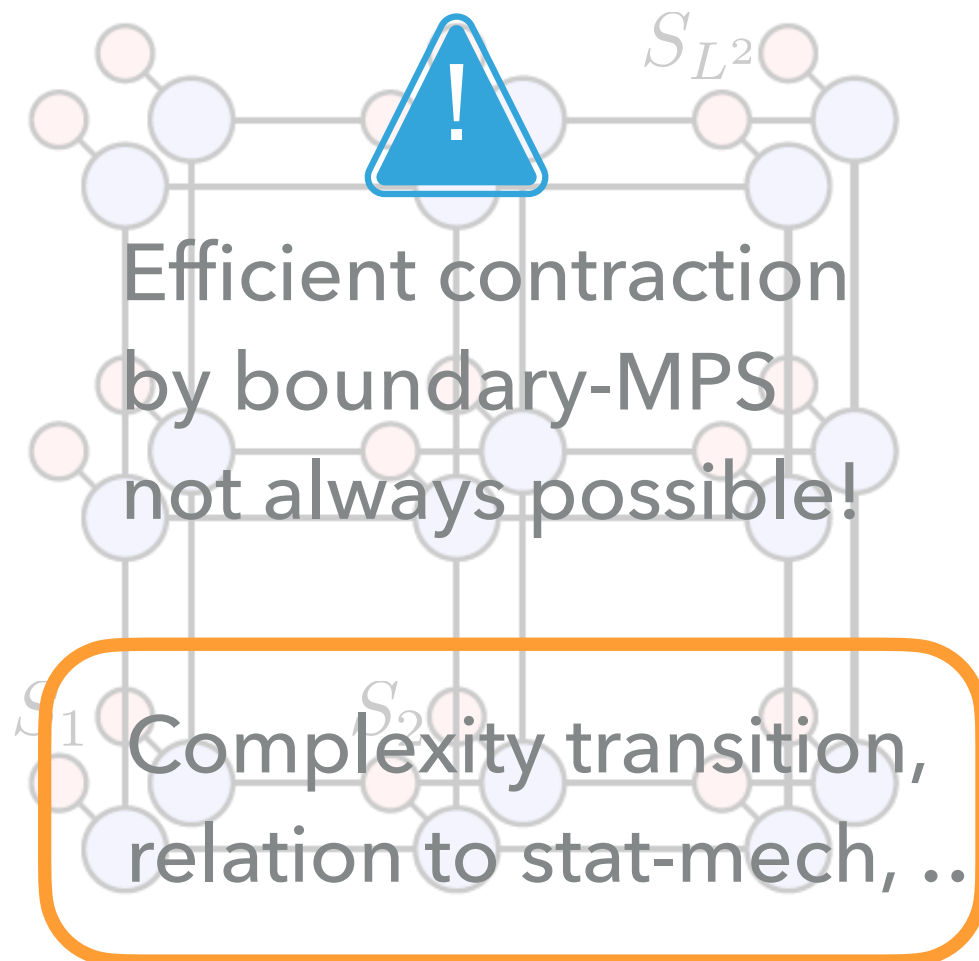
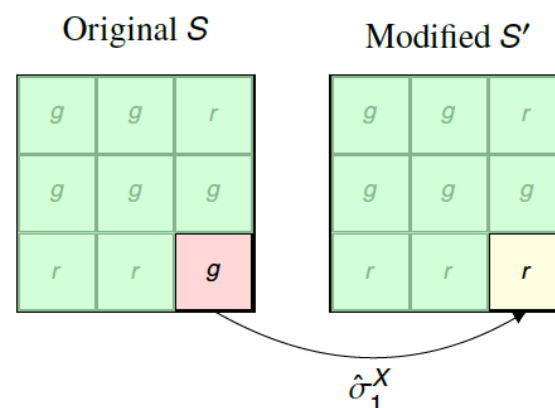
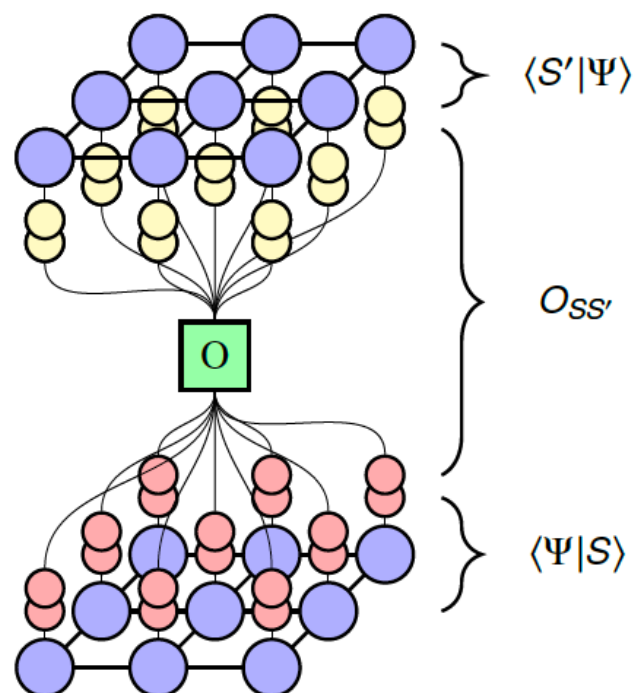
Alcalde, ..., PRB 111, 195120 ('25)

FINITE-PEPS: SAMPLING, ISSUES & FEATURES

Vieijra, et al., PRB 104, 235141 ('21)

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**No autocorrelation
& parallelizable !!!**



Vasseur, ..., PRB 100, 134203 ('19)

Chen, ..., PRX-Q 6, 010312 ('25)

McGinley, ..., PRX 15, 021059 ('25)

Alcalde, ..., PRB 111, 195120 ('25)

**Rydberg interactions $\sigma_j^z \sigma_{j'}^z$
at no extra cost !!!**

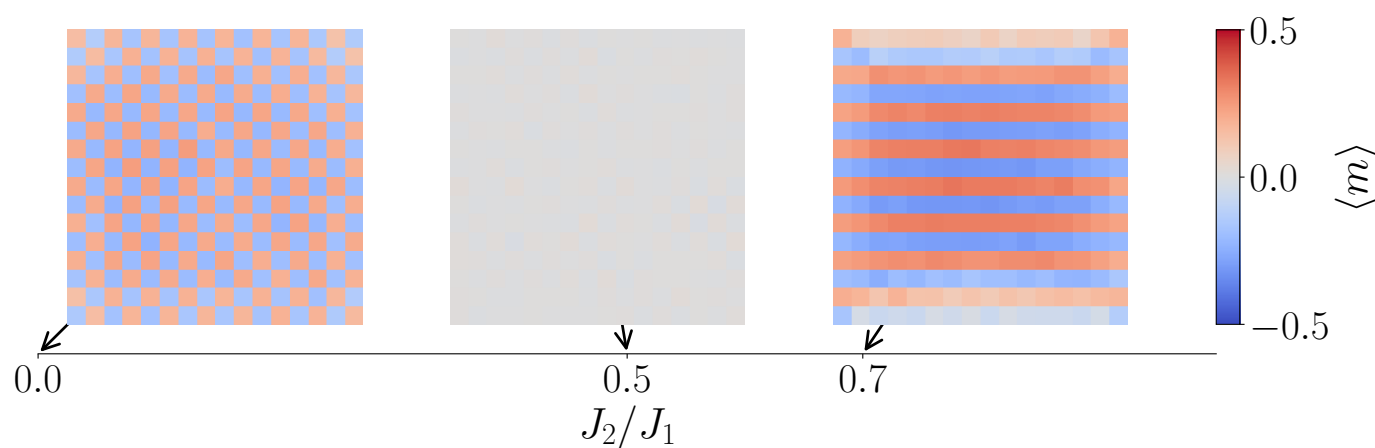
FINITE-PEPS: FIRST BENCHMARKS

52

Alcalde Puente, et al. (MR), PRB 111, 195120 ('25)

Frustrated $J_1 - J_2$ model on square lattice

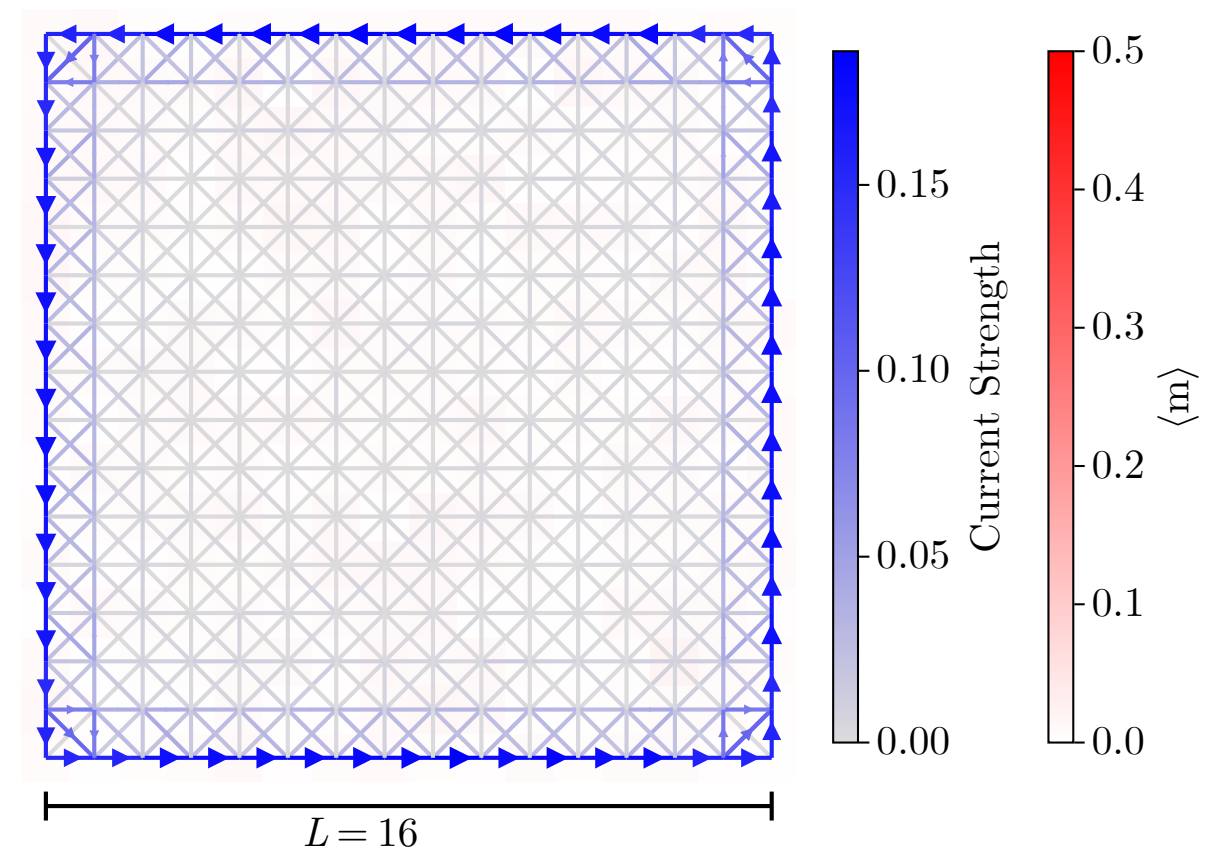
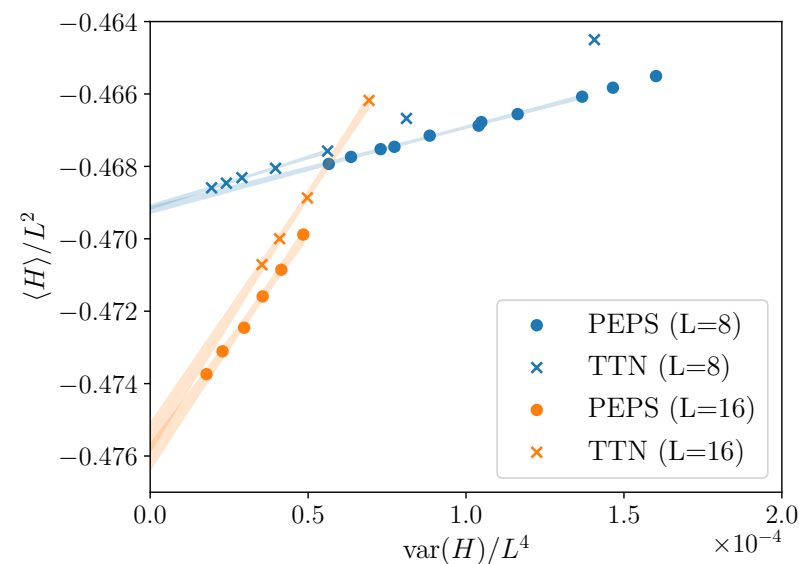
$$\hat{H}_{\text{frust}} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i \cdot \vec{S}_j$$



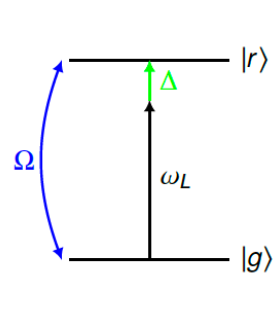
Additional 4-body cyclic term to stabilise a **chiral spin liquid**

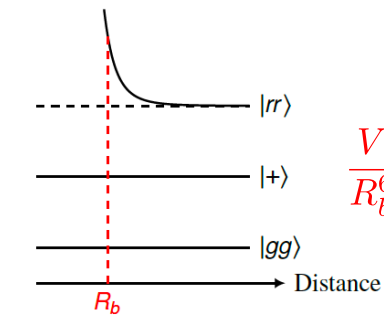
$$\hat{H}_{\text{CSL}} = \hat{H}_{\text{frust}} + i\lambda \sum_{\square} (P_{ijkl} - P_{ijkl}^{-1})$$

Systematic extrapolation of energy estimates & errors



Schröder, et al. (MR), in preparation



$$\hat{H} = \sum_i \left(\frac{\Omega}{2} \hat{\sigma}_i^x - \Delta \hat{n}_i \right) + \frac{1}{2} \sum_{i \neq j} \frac{V}{|\mathbf{x}_i - \mathbf{x}_j|^6} \hat{n}_i \hat{n}_j$$


$$\frac{V}{R_b^6} = \Omega$$

- highly-tunable **quantum simulators**
- predicted to realize **exotic phases** (*frustration, topol. order, gauge th.,...*)
- used to realize **hi-fi qubits & QEC**



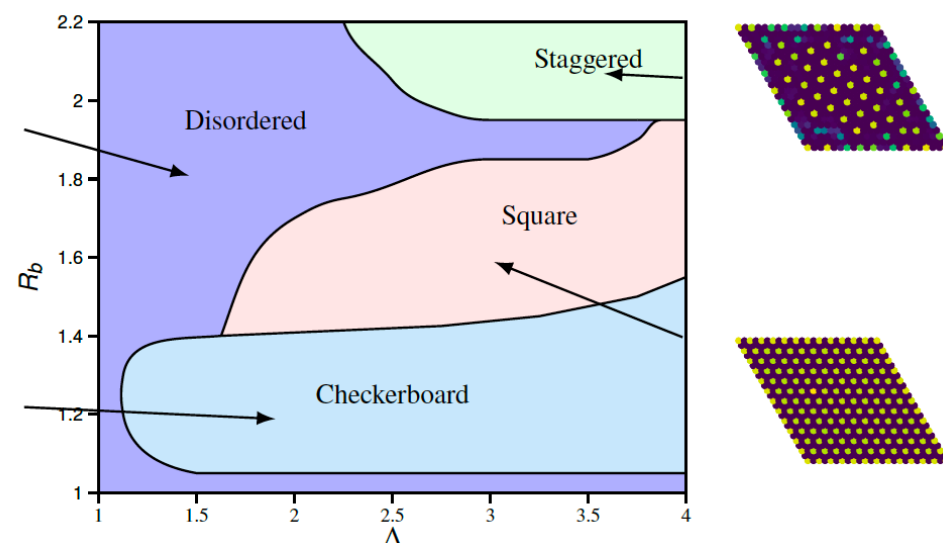
Subtle role of:

- truncation of long-range terms
- auto-correlation of samples

e.g., spin-glass on Kagome lattice!?

Yan et al., PRL 130, 206501 ('23)

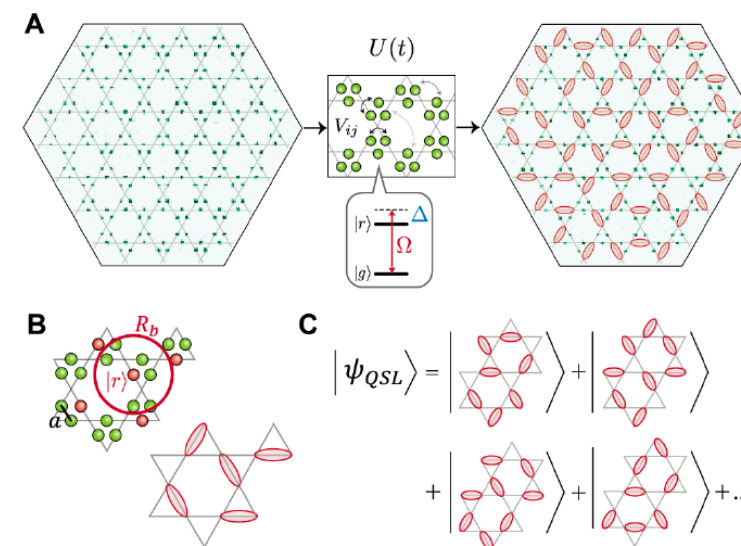
Hibat-Allah, et al., Comm. Ph. 8, 308 ('25)



e.g., spin-liquid on Ruby lattice !?

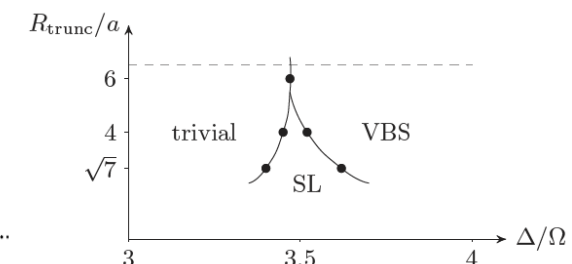
Verresen et al. Phys. Rev. X 11, 031005 ('21)

Semeghini, et al., Science 374, 1242 ('21)



'PXP'-model

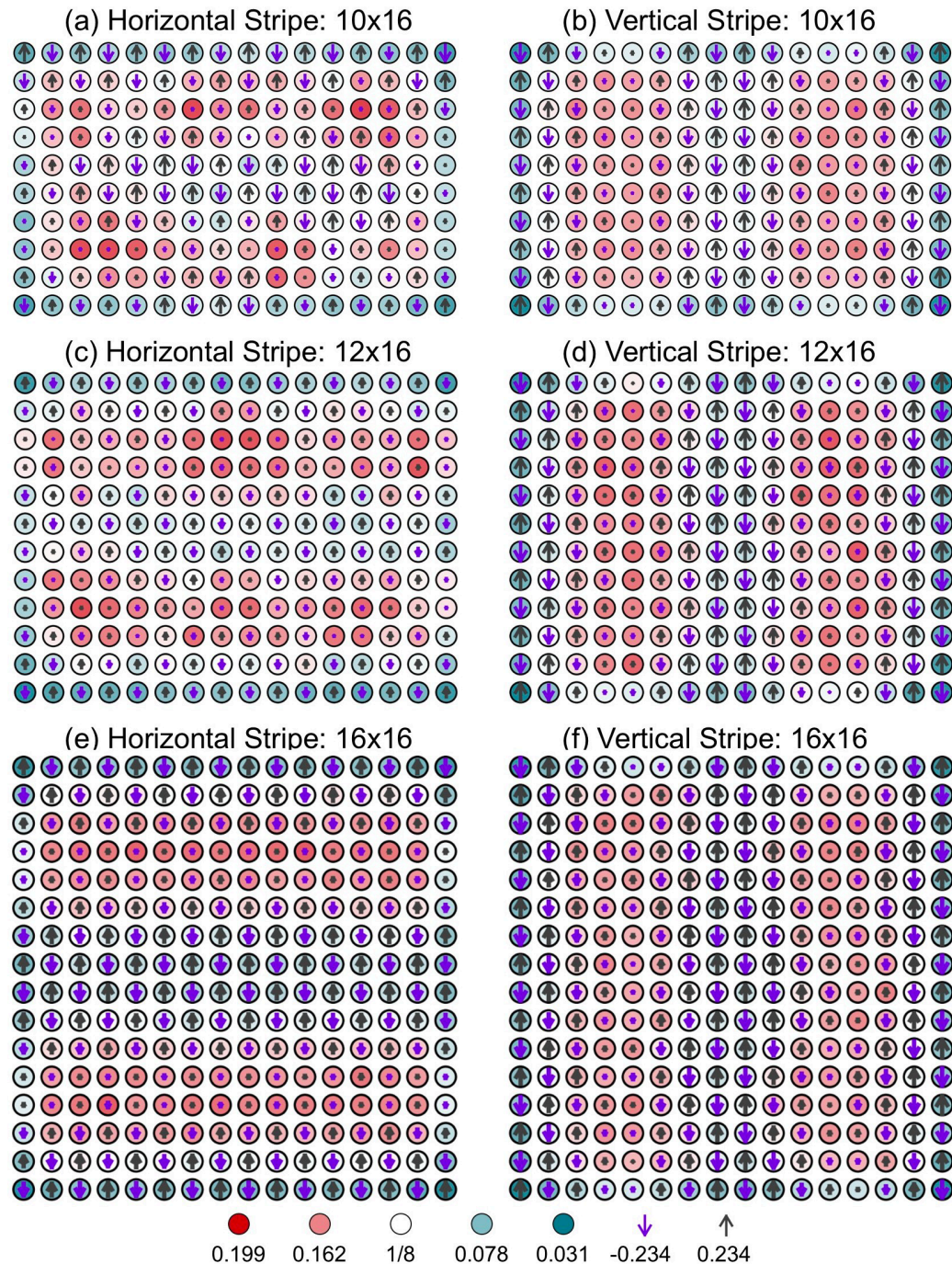
$$V(r) = \begin{cases} +\infty, & r \leq 2a \\ 0, & r > 2a \end{cases}$$



FINITE-PEPS: OTHER COOL APPLICATIONS

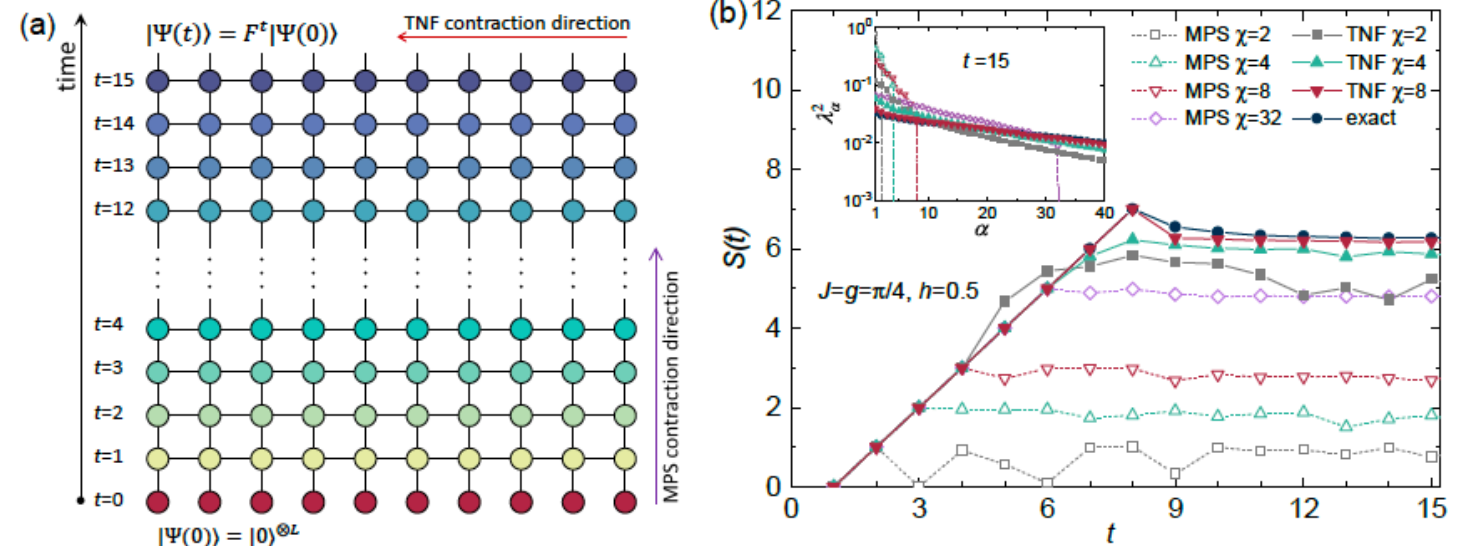
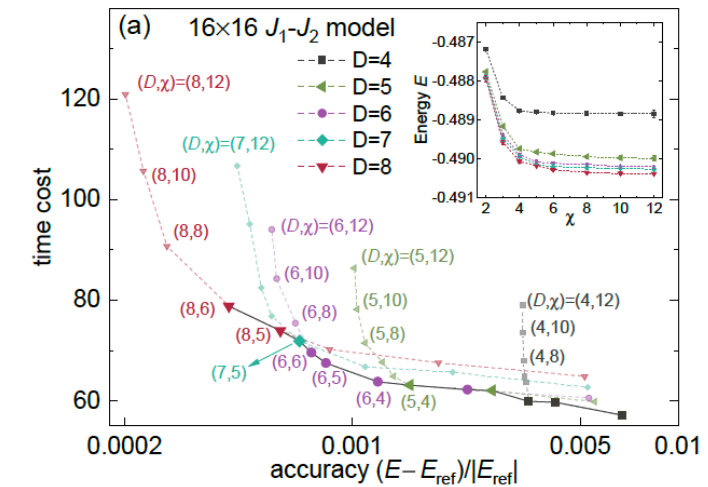
Liu et al., PRL 134, 256502 (2025)

stripes in the 1/8 doped Hubbard model



Liu et al., PRL 133, 260404 (2024)

strict variationality & volume-law



+ efficient representation of NQS

~~Imaginary~~ **TDVP** (a.k.a. **natural gradient**) for N_P param. θ (tensor entries)
Real

$$|\dot{\Psi}(\theta)\rangle = -\dot{\mathbf{i}}H|\Psi(\theta)\rangle \longrightarrow \sum_i \underbrace{\frac{\partial \langle S|\Psi(\theta)\rangle}{\partial \theta_i}}_{\tilde{O}_{S,I}} \dot{\theta}_i = - \underbrace{\langle S|H|\Psi(\theta)\rangle}_{\tilde{E}_S^{\text{loc}}}, \quad \forall S \in \mathcal{H}$$

NQS use this for real-time evolution...

Schmitt, Heyl,
PRL 125, 100503 ('20)
& many more!

...why don't we try for finite-PEPS as well?



? How expensive is to get an accurate gradient? ?
(*imaginary evolution is way more permissive...*)

CONCLUSION

*"Never give up on a dream just because of
the time it will take to accomplish it.
The time will pass anyway."*

[Earl Nightingale]

SPECIAL THANKS TO...

57



E. Weerda



D. Alcalde



N. Tausendpfund



P. Geraghty



J. Nguyen



A. Dubey



K. Schröder



N. Dannenfeld



N. Elbracht



L. Reich



Markus Schmitt



W. Krinitsin

Uni-Regensburg/FZJ

... and to all other collaborators, too!
(J. Naumann, P. Schmoll, J. Eisert, S. Montangero, M. Heyl, ...)

- * Overview of TN approaches for 2D
- * Tree Tensor Networks (TTN):
 - * Basic definitions & operations
 - * App 1: statics of (bosonic) fractional quantum Hall
 - * App 2: dynamics of interfaces (2D Ising)
 - * other applications & extensions
- * Projected Entangled Pair States (PEPS):
 - * Variational algorithm for infinite systems
 - * App: statics of (bosonic) fractional quantum Hall
 - * Stochastic reconfiguration for finite systems
- * Conclusion