Introduction

In this lecture, we will show that:

- Quantum circuits can be described in terms of tensor networks.
- Tensor networks provide a common language for classical and quantum algorithms.
- In certain cases, the use of a quantum computer may allow for exponential speedups in tensor network contraction

MPS and canonical form revisited

Let us start by reminding ourselves of the general ideas of MPS, the diagramatic notation, and canonical form. For concreteness we will consider only spin-1/2 degress of freedom.

Main idea is to represent / approximate the many-body quantum states by the product of tensors (matrices in 1D). That is, for a quantum state of N spin-1/2,

$$|\psi\rangle = \sum_{ij\cdots k=\uparrow,\downarrow} \psi_{ij\cdots k} \, |ij\cdots k\rangle ,$$

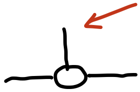
we can write the probabilty amplitudes as

$\psi_{ij\cdots k}$ =

may be an equality
for particular states or
for certain bond dimension χ

The state is now written as a product of tensors

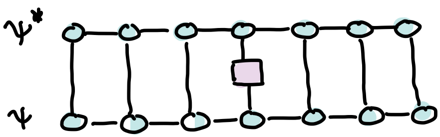
physical index
dimension $d_i = 2$

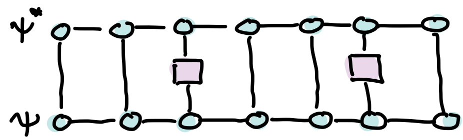


virtual indices
dimension χ_i, χ_{i+1}

Given the MPS, our goal is typically to compute local observables that correspond to measurable quantities, e.g.

$\langle \psi | \sigma^z_i | \psi \rangle$ or $\langle \psi | \sigma^x_i \sigma^x_j | \psi \rangle$





And the beauty of using MPS is that the computational cost is polynomial in both the number of sites N and the bond dimension χ_{max}, in contrast to the brute-force exponential cost in N.

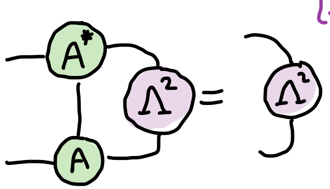
Canonical form & Isometric form

There is a gauge freedom in how we specify the tensors in the MPS, but there are particularly useful canonical forms. These have two conditions

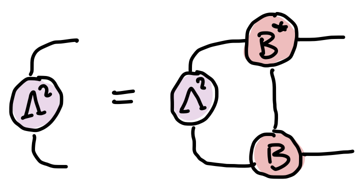
Left canonical **Right canonical**

(1)

(2)



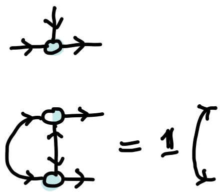
$[\Lambda]_{ij} = \lambda_i \delta_{ij}$ the Schmidt values.

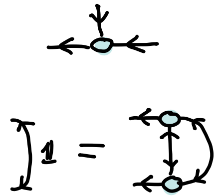


If we relax the second condition then we get the slightly more general isometric form. To deal with the isometric form we can introduce a new arrow notation, namely

<u>left isometric</u> <u>right isometric</u>

(Isometry condition)





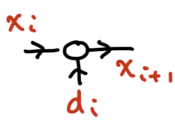
The convention of the arrows is such that the total dimension of ingoing arrows $\geq$ outgoing arrows.

By grouping legs, we can talk about isometric matrices

> Let A be a $M \times N$ matrix with $M \geq N$
> then A is an isometry iff
> $$A^{+}A = \mathbb{1}_{N\times N} \quad \text{and} \quad AA^{+} = P_{M\times M}$$
> where $P_{M\times M}$ is a projector ($P^2 = P$ with $\mathrm{rank}(P) \leq N$)

Note for an MPS

$$d_i \chi_i \geq \chi_{i+1}$$



Similarly $\chi_i \leq d_i \chi_{i+1}$

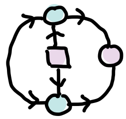
Also note that that we assume.

$\mathrm{rank}(P) = N$

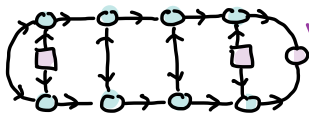
for MPS, otherwise there is redundancy in the MPS.

Canonical form and isometric form allow us to more efficiently compute observables.

E.g.

$\langle\psi|\sigma^z_i|\psi\rangle =$ 

Λ^2 for canonical

$\langle\psi|\sigma^x_i\sigma^x_j|\psi\rangle =$ 

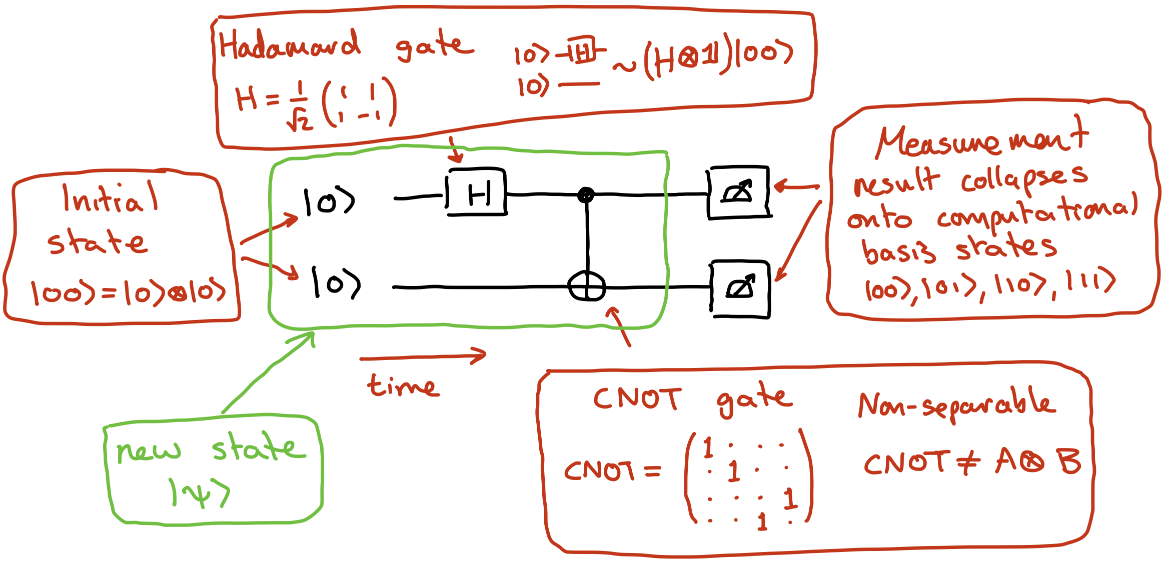
Quantum Circuits

Quantum circuits are the leading model for quantum computing. Just like with tensor networks, it is typical to work with diagrams. Typically they consist of three parts:

- an initial state, normally the product state
  $$|000\cdots00\rangle \sim |\uparrow\uparrow\uparrow\cdots\uparrow\uparrow\rangle$$
- unitary gates (operators)
- Measurements.

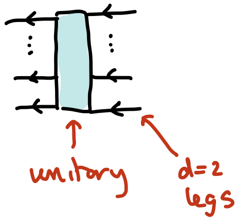
Note we typically deal with qubits where $|0\rangle \sim |\uparrow\rangle$, $|1\rangle \sim |\downarrow\rangle$

An example quantum circuit that creates and measures the Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

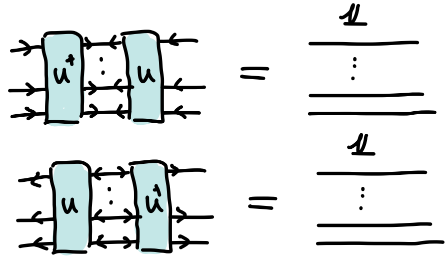


For our purposes we will ignore the subtleties with measurement. Initial state + unitary gates = new state. Assume we can extract $\langle\psi|\sigma^z_i|\psi\rangle$ or $\langle\psi|\sigma^x_i\sigma^x_j|\psi\rangle$

In anticipation, let us denote a general unitary using arrow notation, ie.

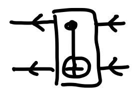


where unitarity is



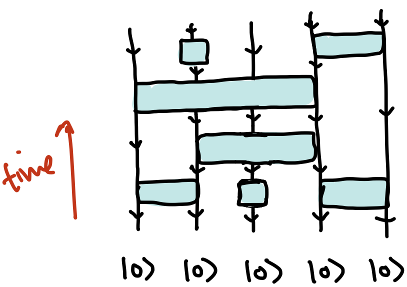
Note that a unitary is an isometry where the incoming and outgoing dimensions match.

- Hadamard had 1 ingoing/outgoing 

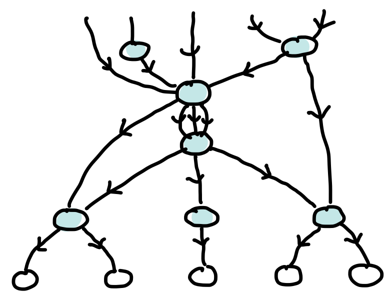
- CNOT had 2 ingoing/outgoing 

More generally, a quantum circuit is of the form

Quantum circuit



tensor network



Unitary circuit is an tensor network in isometric form where unitary gates have equal number of incoming and outgoing legs.

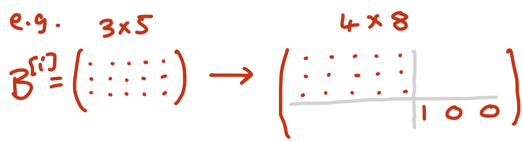
Note: I will typically draw my unitary circuits with time going vertically, which is not standard!

Mapping MPS to sequential quantum circuits

We can do better than simply noting that we can interpret quantum circuits as tensor networks.
There is an exact mapping between MPS and certain sequential quantum circuits.

Let us start with an MPS in left canonical form

And consider a single isometric tensor. We will consider bond dimensions χ_i, χ_{i+1} that are powers of 2 for simplicity.

> Note: We can embed a tensor in one of the power of 2 form. We embed each matrix as the upper left block and place a maximal rank projector in the bottom right block.
>
> e.g.
>
> 
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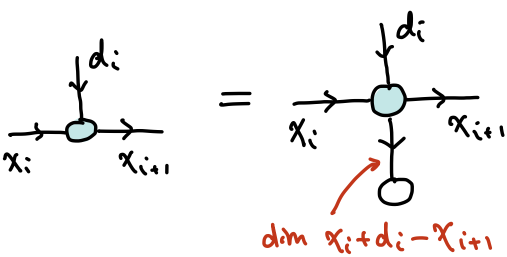
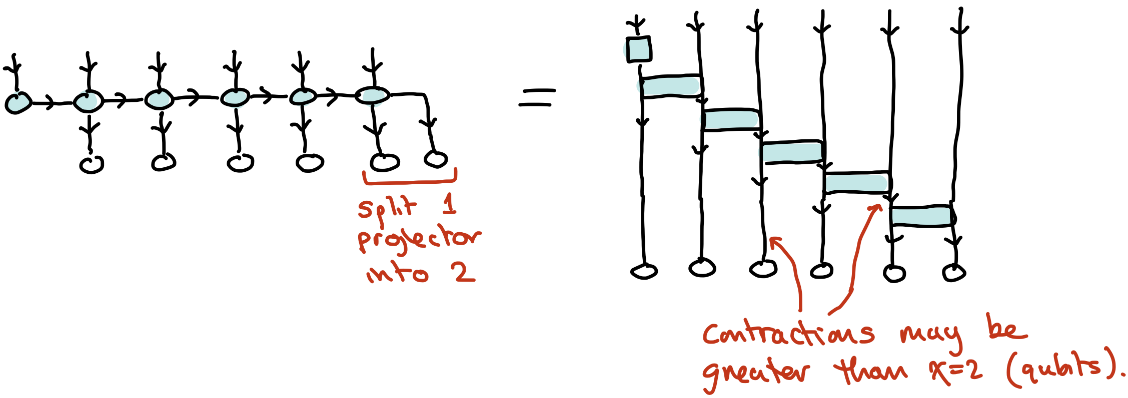
We can then promote these isometries to unitaries + projectors

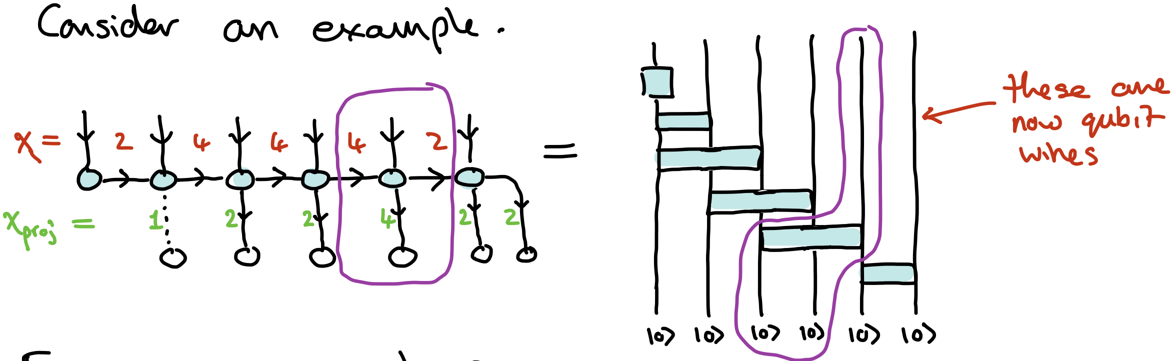


We can choose the projector w.l.g. to be $[1,0,0,\cdots]$, and then use QR-decomposition to find the corresponding unitary. It will be easiest to demonstrate this when we work through an explicit example.

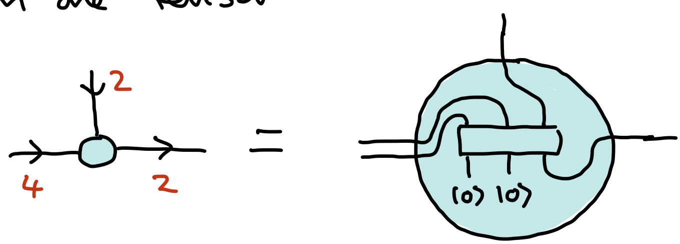
The MPS can then be written as



Consider an example.



Focussing on one tensor

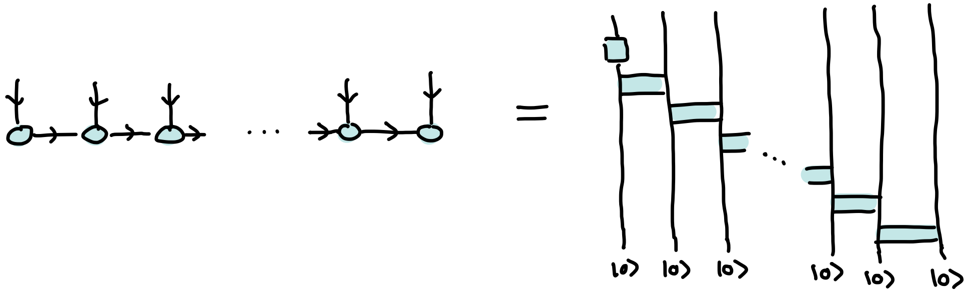


Explicit Example : GHZ state

The GHZ state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\downarrow\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(|00\cdots0\rangle + |11\cdots1\rangle\right)$$

Is exactly represented by a $\chi = 2$ MPS



Where the MPS tensors are

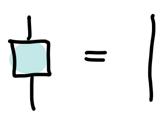
$$A^{[1]\uparrow} = (1,0) \qquad A^{[i]\uparrow} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A^{[N]\uparrow} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A^{[1]\downarrow} = (0,1) \qquad A^{[i]\downarrow} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^{[N]\downarrow} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

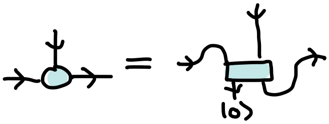
The easiest unitary to find corresponds to $A^{[1]}$ since this is already unitary

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^{[1]\uparrow} = (1,0) \qquad A^{[1]\downarrow} = (0,1)$$

the first unitary is the identity



Let's next consider the unitary corresponding to $A^{[i]}$



$$U = \begin{pmatrix} 1 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot \end{pmatrix}$$

$$A^{[i]\uparrow} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A^{[i]\downarrow} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

↑ need to find two columns such that all columns are orthonormal

We can find missing columns by Gram-Schmidt.
Equivalently, randomly fill columns and use QR-decomposition

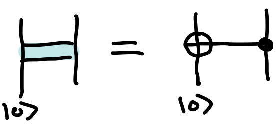
$M = QR$, Q-unitary, R- upper triangular.

The choice of unitary is <u>not unique</u>!

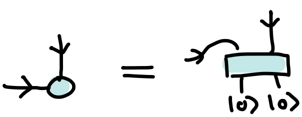
One choice that works is

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = CNOT_{21}$$

i.e.



The final unitary to find corresponds to $A^{[N]}$.



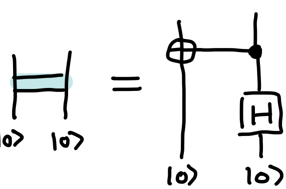
$$U = \begin{pmatrix} 1/\sqrt{2} & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 1/\sqrt{2} & \cdot & \cdot & \cdot \end{pmatrix}$$

$$A^{[N]\uparrow} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad A^{[N]\downarrow} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

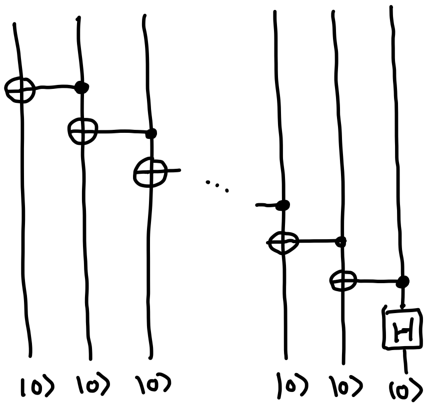
A unitary that works is

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} = CNOT_{21}(\mathbb{1}\otimes H)$$

i.e.

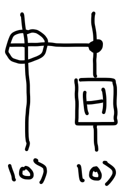


Finally we end up with the quantum circuit



We can easily check that this quantum circuit does indeed prepare the GHZ state.

In fact, our first Bell State circuit was a special case of this circuit with $N=2$



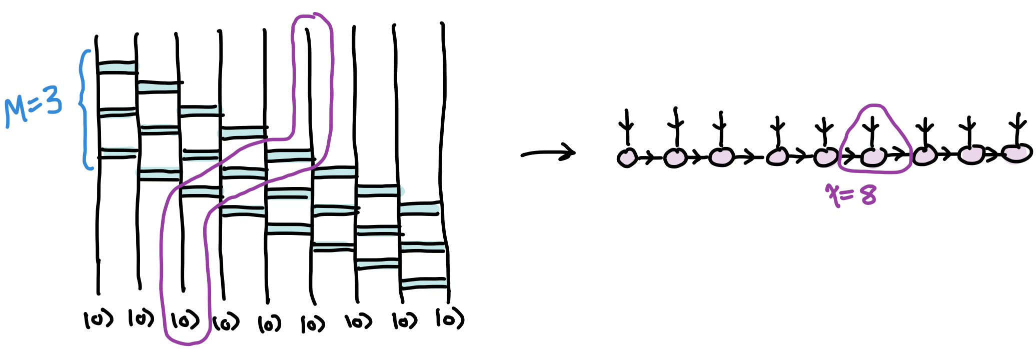
Why Quantum Circuits?

So why have we bothered making this connection between tensor networks and MPS?
How can we use it?

One reason is that is valuable to have a common language. It may allow us to take lessons learnt from MPS and TNS and apply them to quantum circuits, and vice versa.

However, there is a much more direct reason.
A quantum computer can perform certain tensor network contractions exponentially more efficiently than classical computers.

To see this, let's do the reverse of what we did previously and write down a quantum circuit and map back to MPS. Consider the following type of sequential circuit.



The circuit I have drawn is a subset of MPS with $\chi = 8$.

More generally if I use M sequential layers, then this is a subset of MPS with $\chi = 2^M$.

It has $\mathcal{O}(MN)$ number of parameters, whereas the $\chi = 2^M$ MPS has $\mathcal{O}(\chi^2 N) = \mathcal{O}(4^M N)$ parameters.

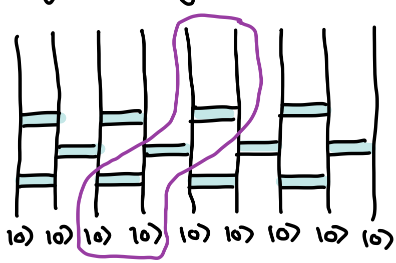
Nevertheless, as long as $N > M$, the cost of classically contracting this network scales with $\chi = 2^M$.

However, on the quantum computer, we need N qubits and a runtime that is proportional to $N+M$. This is an exponential improvement!

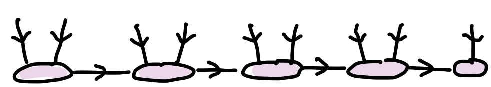
This is a type of "sparse" quantum-MPS that can be efficiently contracted on a quantum computer.

These states are physically relevant for non-equilibrium dynamics.

As a second example consider the shallow brickwall circuits. These are particularly efficient on quantum computers as they use nearest-neighbour 2-qubit gates and are fixed depth

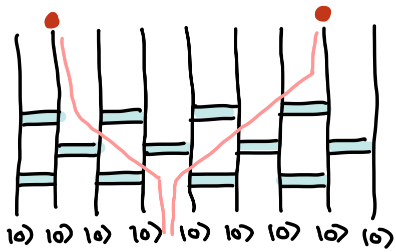


$\longrightarrow$



← Note this is actually a subset of sequential circuits!

By limiting the quantum computing time to $O(M)$, we have introduce an additional restriction on the state.



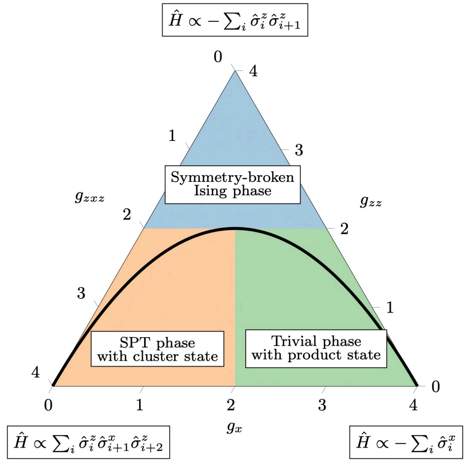
There are no correlations between the red qubits.

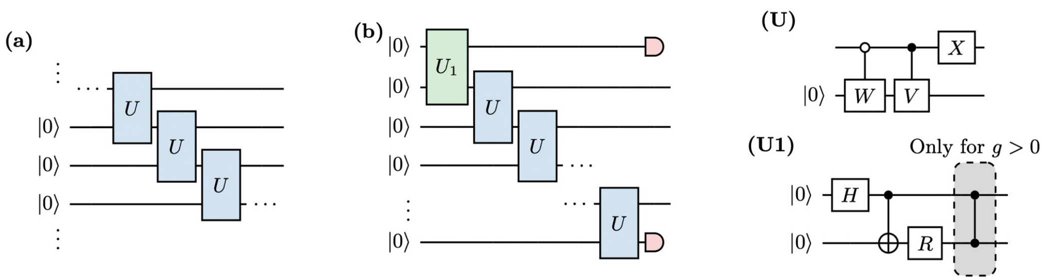
Since we start from a product state, correlations can build up only if the "backwards lightcones" overlap.

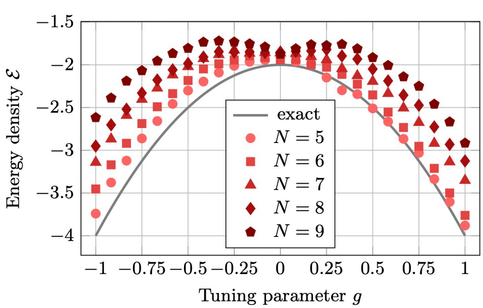
This type of circuit has a strict correlation length instead of the exponential decay of general MPS.

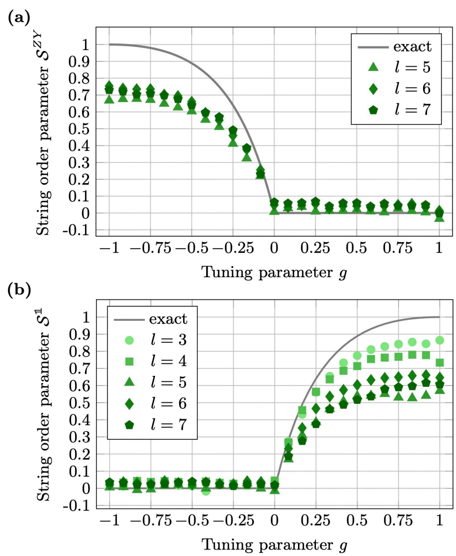
Example: Crossing a topological phase transition

Adam Smith, Bernhard Jobst, Andrew G. Green, and Frank Pollmann [Phys. Rev. Research 4, L022020 (2022)]



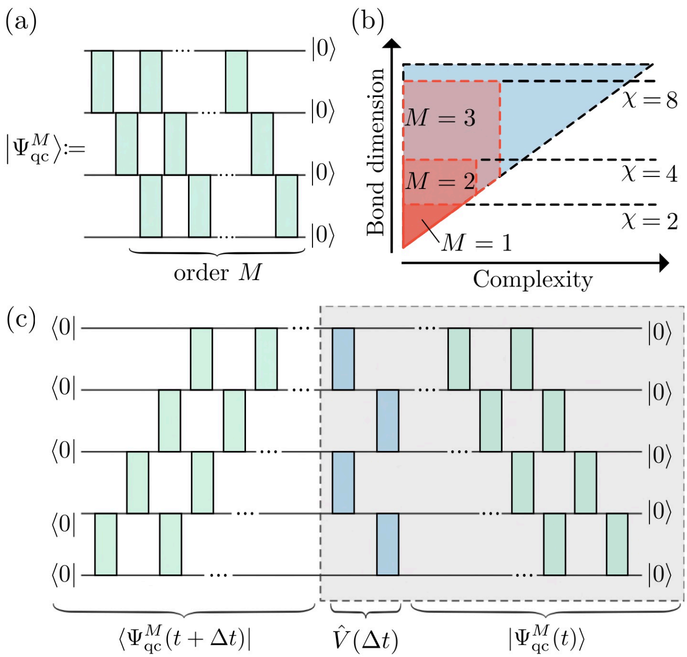


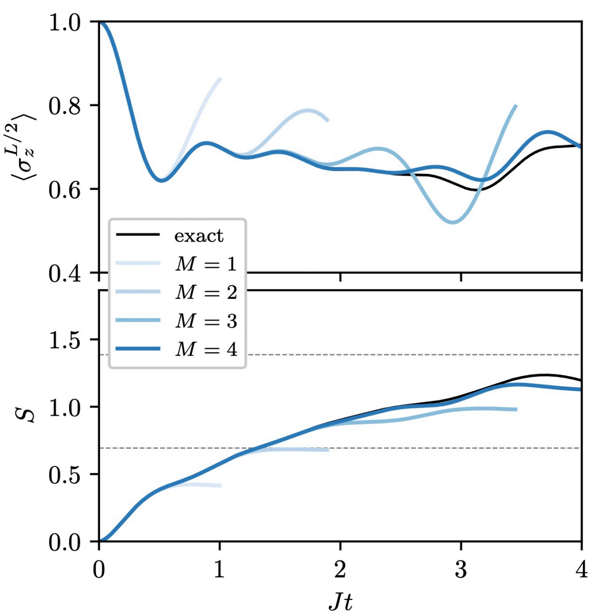


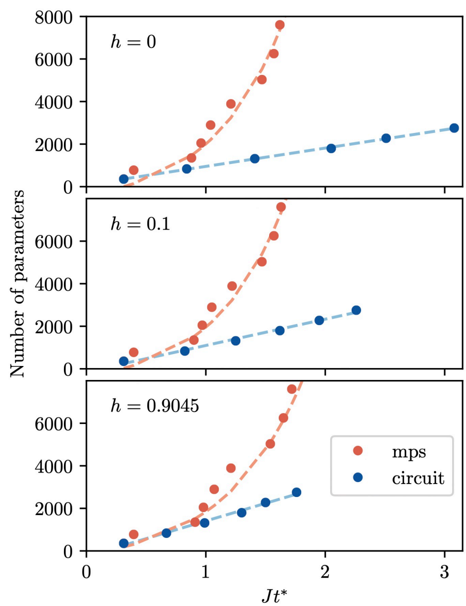


Example: Quantum-TEBD

Sheng-Hsuan Lin, Rohit Dilip, Andrew G. Green, Adam Smith, Frank Pollmann [PRX Quantum 2, 010342 (2021)]

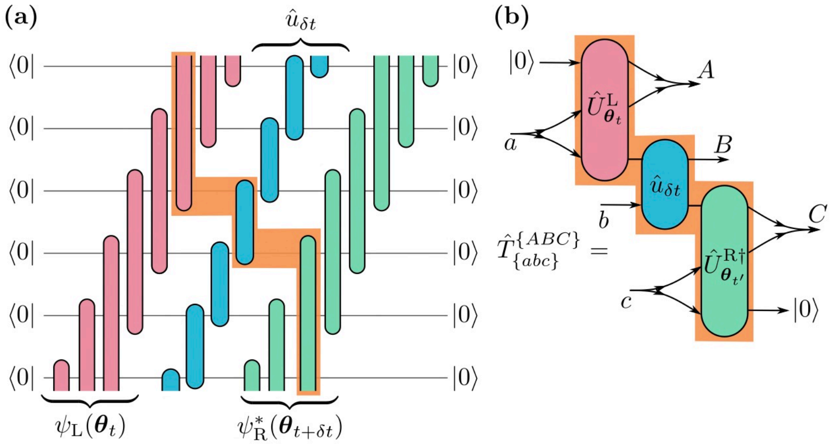


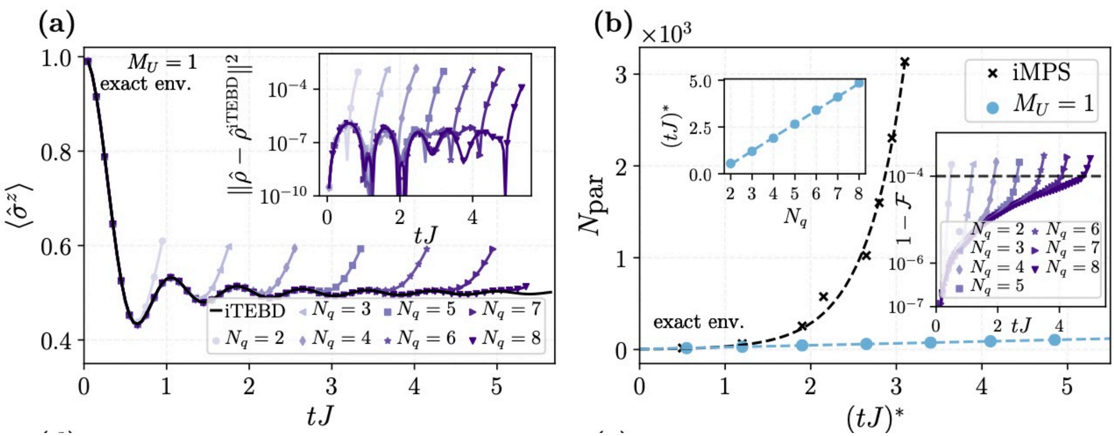




Example: Infinite Quantum-TEBD

Nikita Astrakhantsev, Sheng-Hsuan Lin, Frank Pollmann, Adam Smith [arXiv:2210.03751]





References

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Variational power of quantum-MPS

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Fergus Barratt, James Dborin, Matthias Bal, Vid Stojevic, Frank Pollmann, Andrew G. Green [npj Quantum Inf 7, 79 (2021)]

IBM experiment using infinite quantum-MPS across a topological phase transition

Adam Smith, Bernhard Jobst, Andrew G. Green, and Frank Pollmann [Phys. Rev. Research 4, L022020 (2022)]

Quantum version of TEBD

Sheng-Hsuan Lin, Rohit Dilip, Andrew G. Green, Adam Smith, Frank Pollmann [PRX Quantum 2, 010342 (2021)]

Quantum version of iTEBD

Nikita Astrakhantsev, Sheng-Hsuan Lin, Frank Pollmann, Adam Smith [arXiv:2210.03751]

IsoTNS to quantum circuits

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Preparation of matrix product states with log-depth quantum circuits

Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, J. Ignacio Cirac [arXiv:2307.01696]