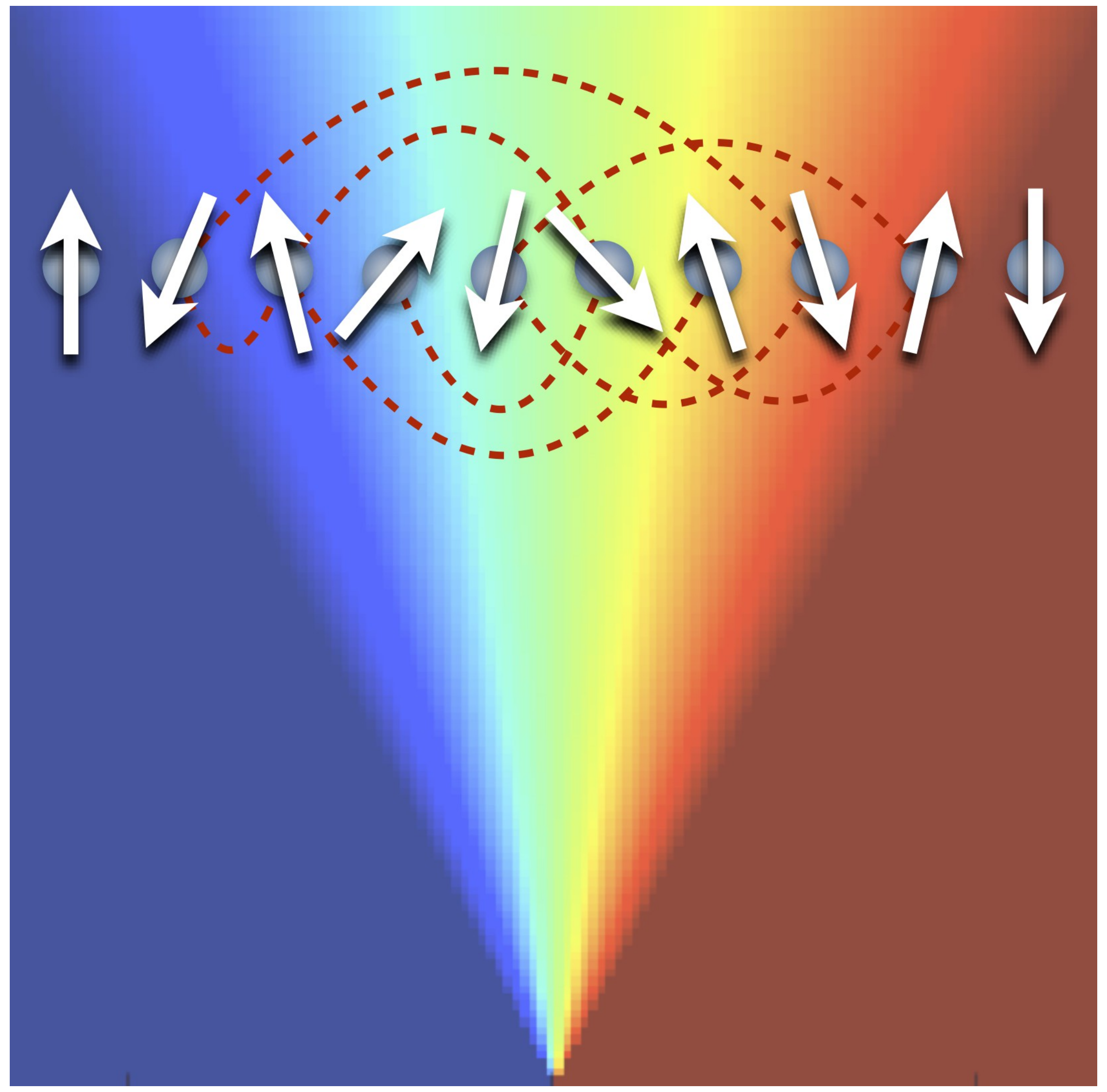
Random unitary circuits as solvable models of generic quantum dynamics



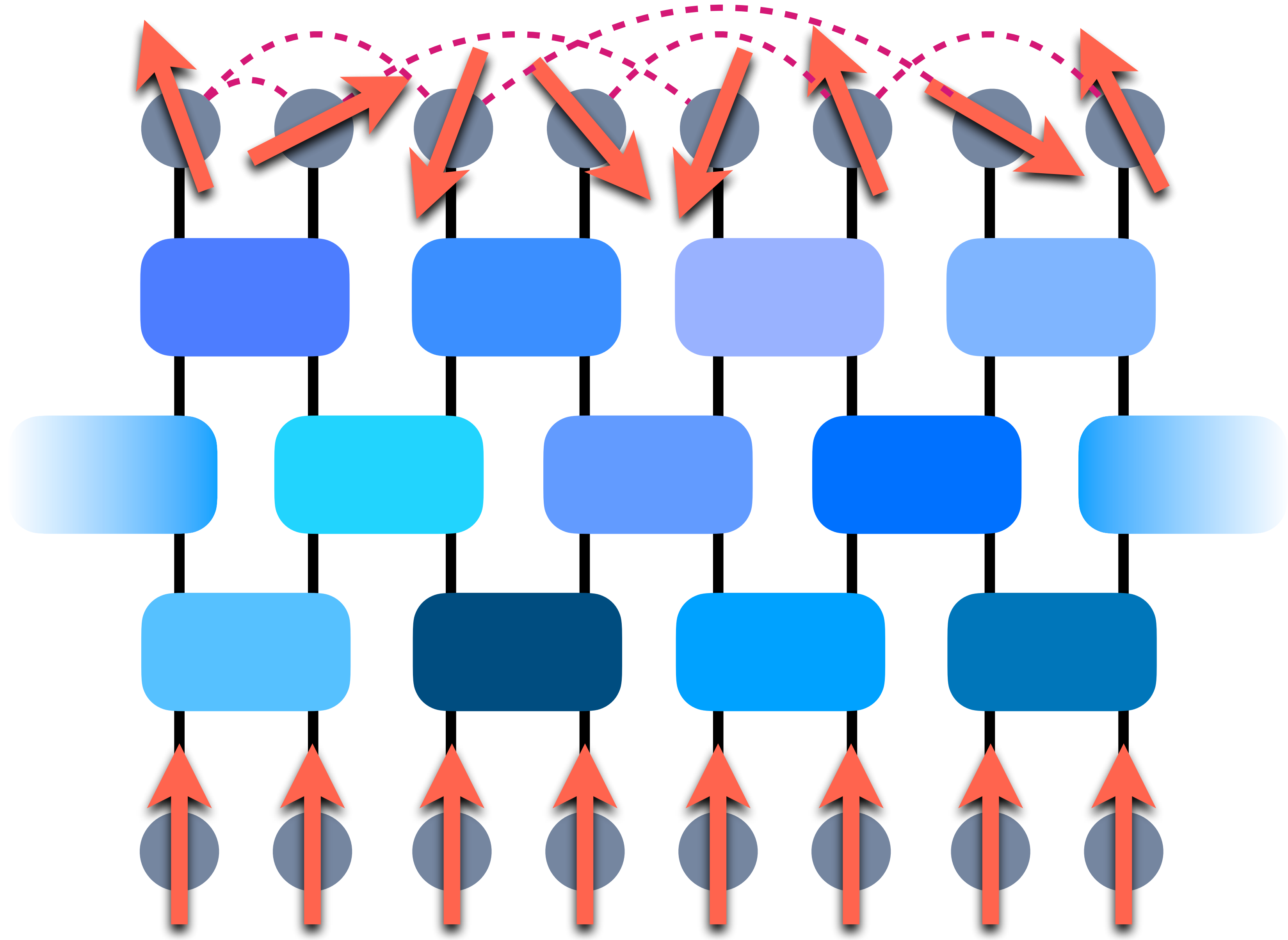
Tibor Rakovszky

MŰEGYETEM 1782

Plan

Introduction: Closed many-body systems far from equilibrium

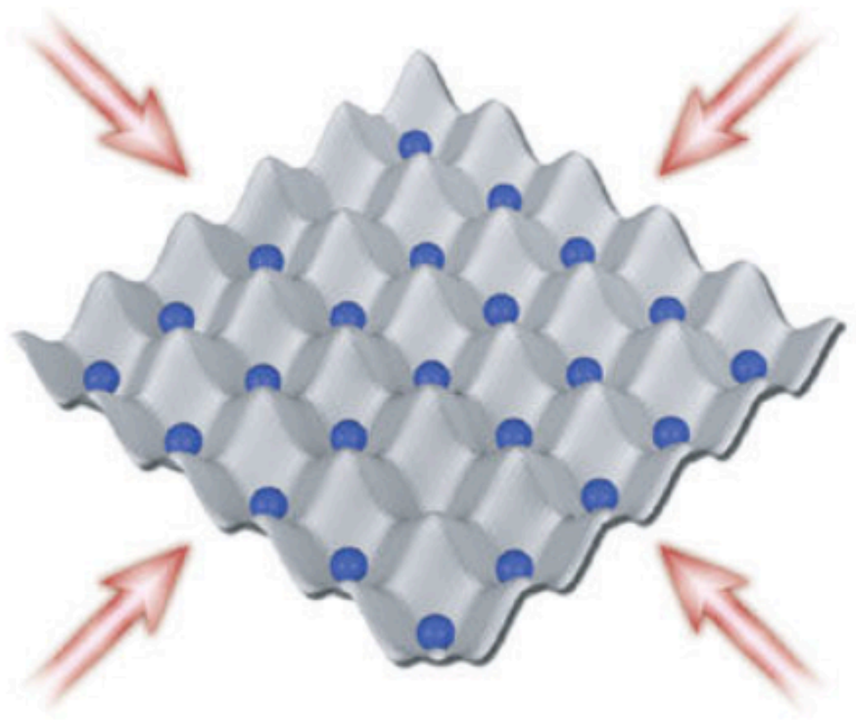
Part 1: Haar random circuits as solvable models



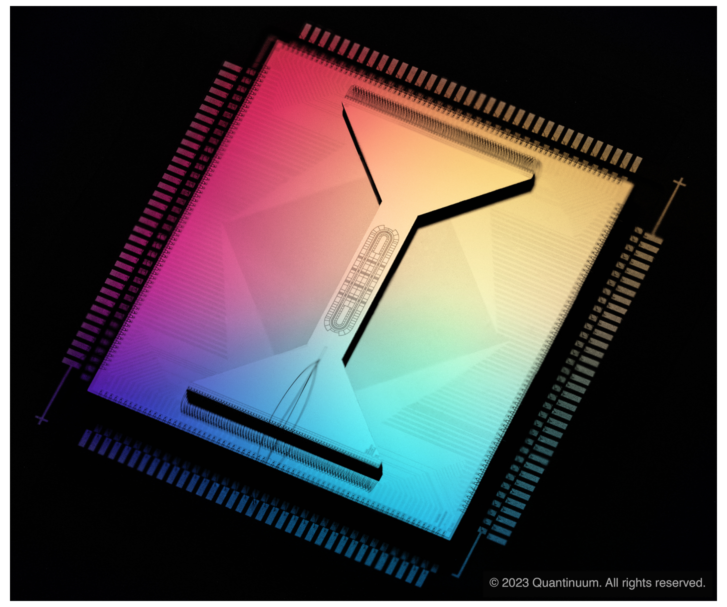
Part 2: Symmetries, measurements and all that

Artificial quantum many-body systems

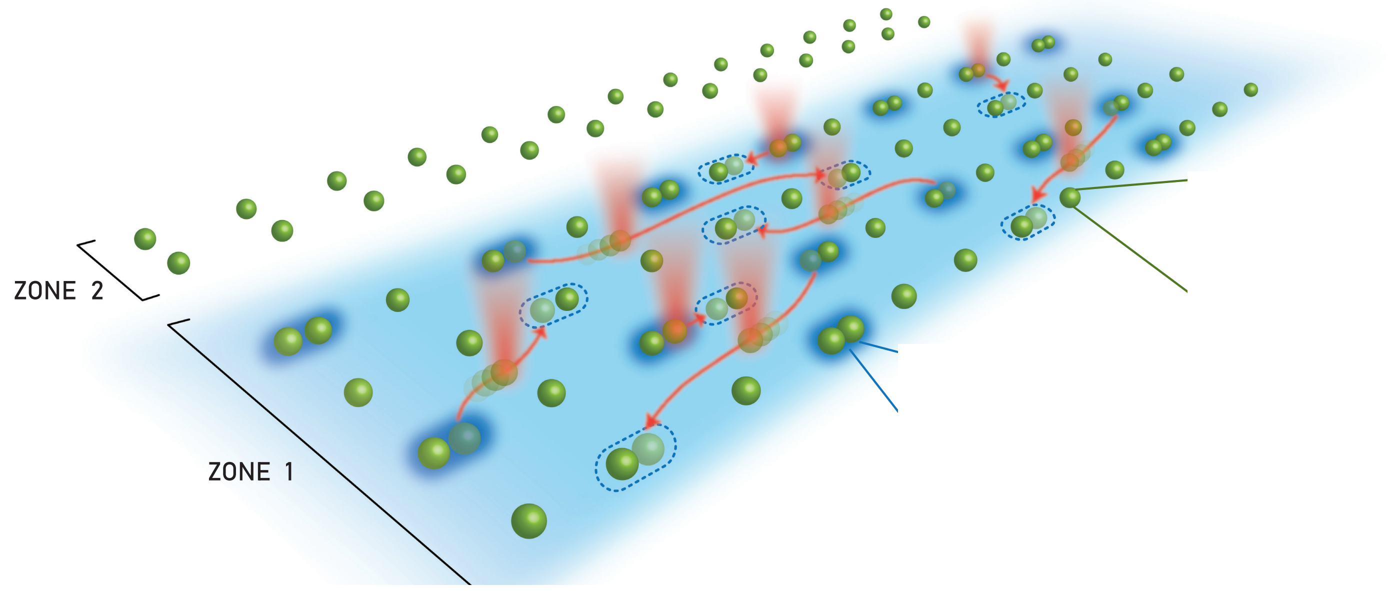
- (Analog) quantum simulators / (Digital) quantum computers
- Well-isolated from environment
- Detailed control over interactions and initial states
- Locally resolved measurements



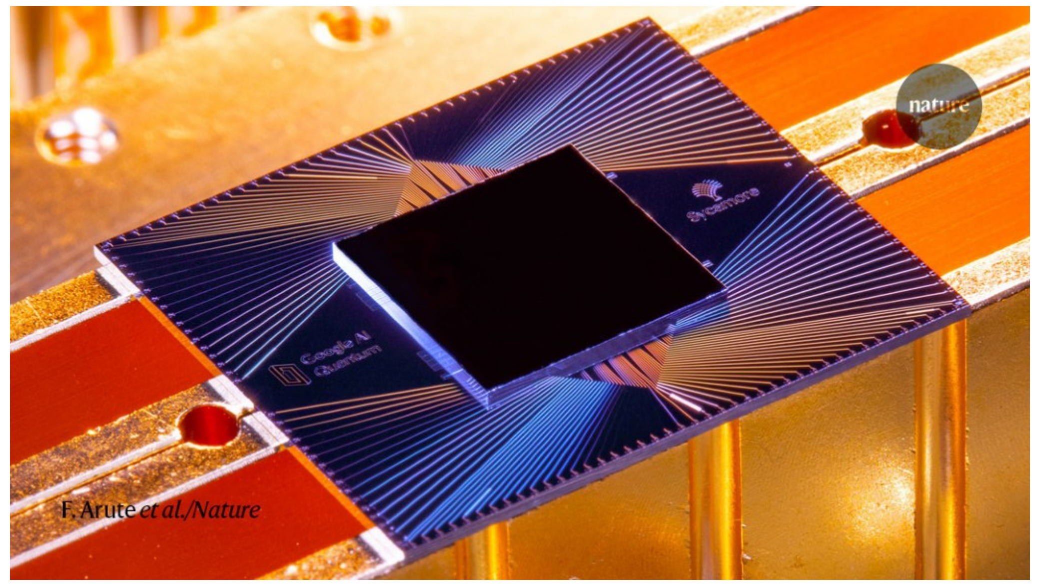
Cold atoms



Trapped ions



Rydberg arrays



Superconducting circuits

Dynamics in closed quantum many-body systems

$$|\Psi(t)\rangle = e^{-i\int \mathrm{d}t\hat{H}(t)}|\Psi(0)\rangle = \hat{U}(t)|\Psi(0)\rangle$$

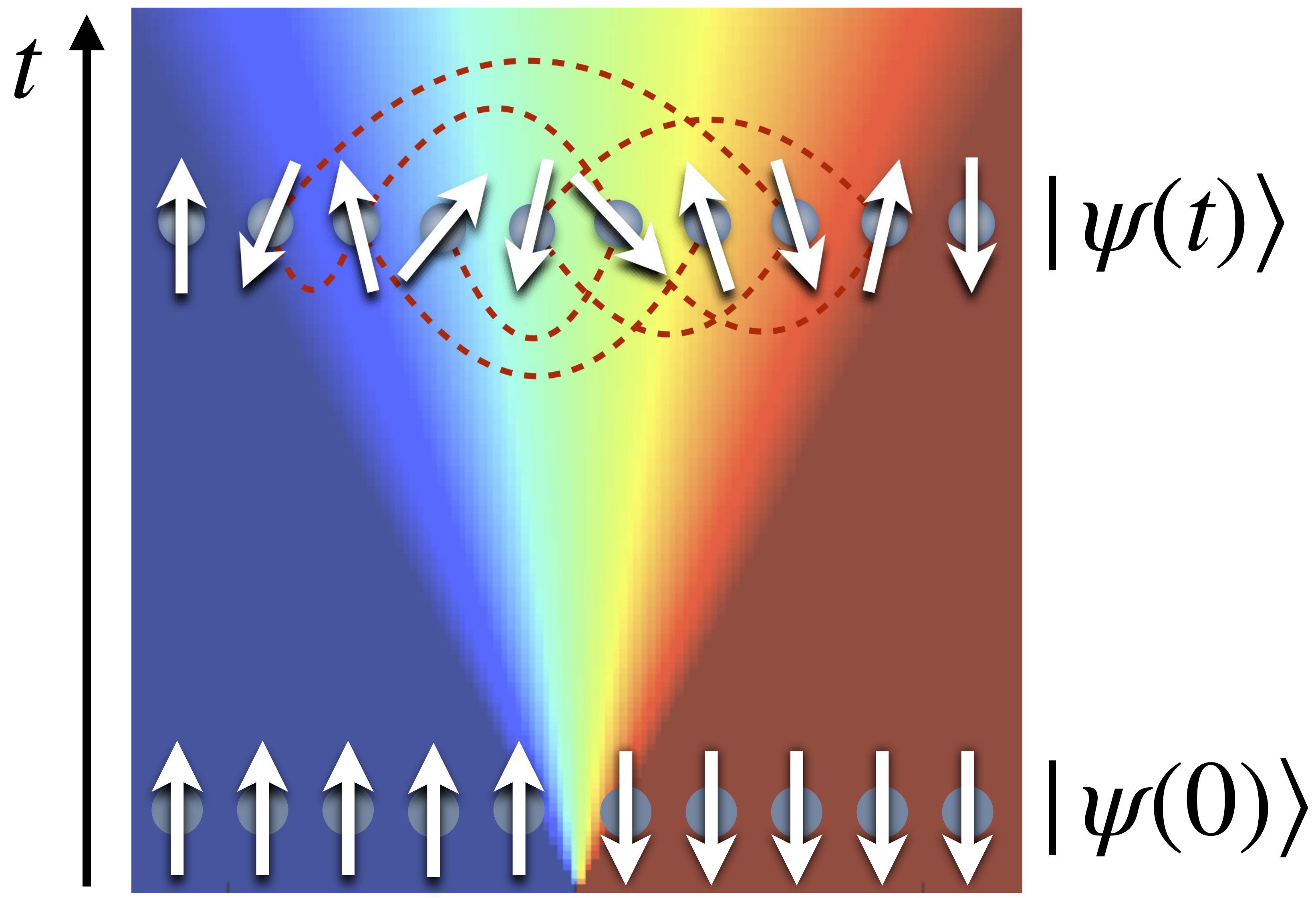
N qubit wavefunction

Evolving with local Hamiltonian

Simple (e.g. uncorrelated) initial state

Quantum correlations are generated dynamically and spread in space

Lieb-Robinson theorem: local interactions lead to emergent "speed of light"



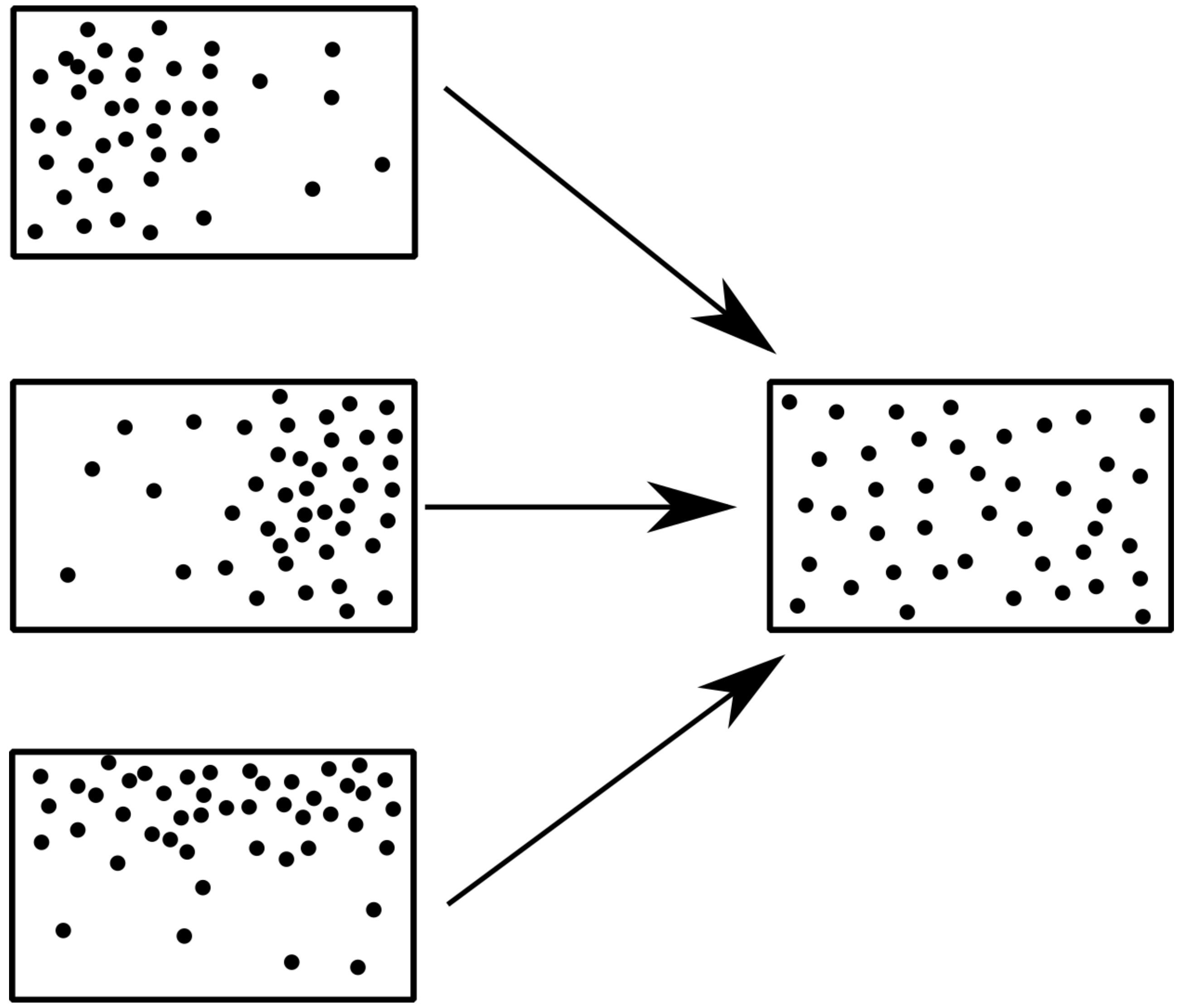
The "paradox" of thermalization

Closed system: $|\Psi(t)\rangle = e^{-i\hat{H}t}|\Psi(0)\rangle$

???

Basic postulate of statistical physics: system eventually reaches thermal equilibrium state

Microcanonical ensemble: $\rho_{\rm mc} \propto \sum_{\alpha: E_\alpha \approx \langle \hat{H} \rangle} |\Psi_\alpha\rangle\langle\Psi_\alpha|$



Irreversible, non-unitary!

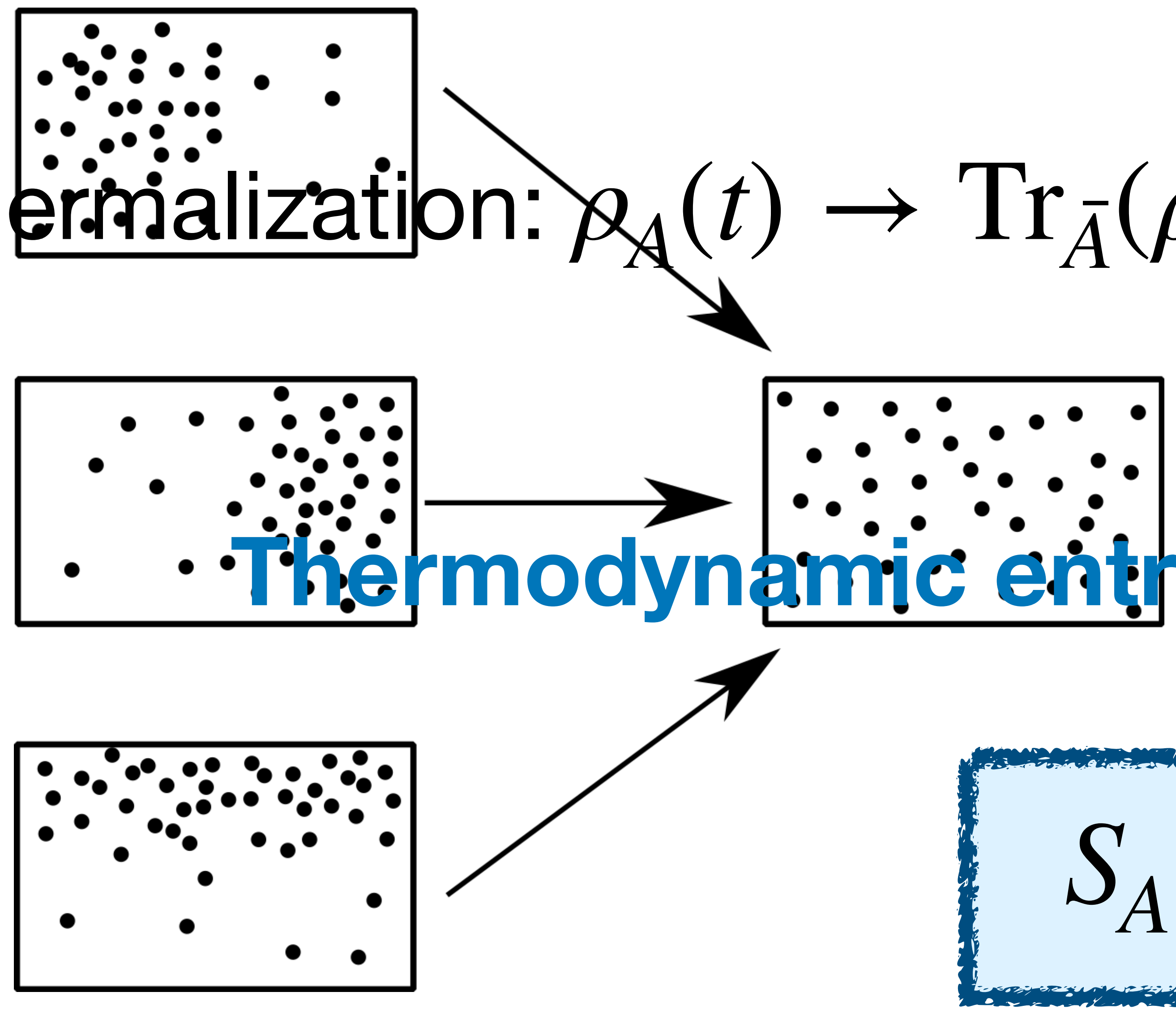
The “paradox” of thermalization… and its resolution

Closed system: $|\Psi(t)\rangle = e^{-i\hat{H}t}|\Psi(0)\rangle$ ↔ ???

Information isn’t lost, but is **delocalized**

Microcanonical ensemble: $\rho_{\mathrm{mc}} \propto \sum_{\alpha: E_\alpha \approx \langle \hat{H} \rangle} |\Psi_\alpha\rangle\langle\Psi_\alpha|$

State of a subsystem: $\rho_A(t) = \mathrm{Tr}_{\bar{A}}(|\Psi(t)\rangle\langle\Psi(t)|)$



Thermalization: $\rho_A(t) \to \mathrm{Tr}_{\bar{A}}(\rho_{\mathrm{mc}})$ if $N_A \ll N$

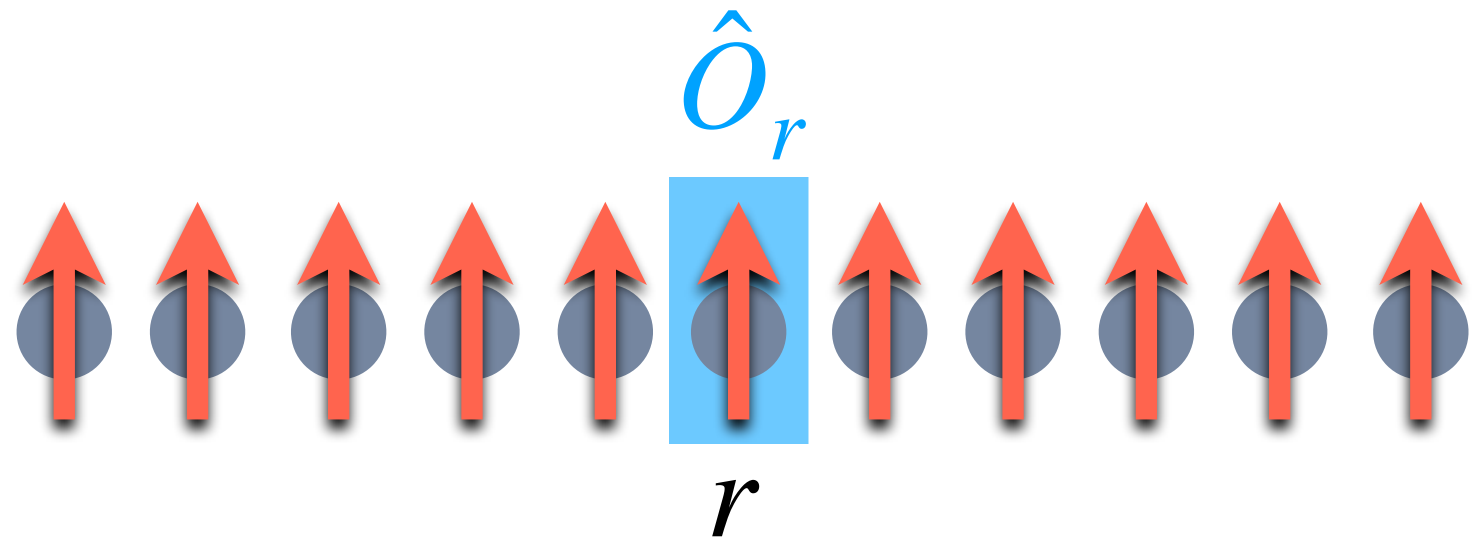
Thermodynamic entropy = Entanglement built up through time evolution

$$S_A(t) = -\mathrm{Tr}(\rho_A \ln \rho_A) \to N_A s_{\mathrm{thermo}}$$

How to characterize the spreading of information?

Schrodinger picture: $|\Psi(t)\rangle = \hat{U}(t)|\Psi\rangle$ → Heisenberg picture: $\hat{O}(t) = \hat{U}(t)^\dagger \hat{O} \hat{U}(t)$

Let $\hat{O} = \hat{O}_r$ be an operator acting at location r (e.g.. $\hat{O}_r = \hat{\sigma}^z_r$ spin operator)



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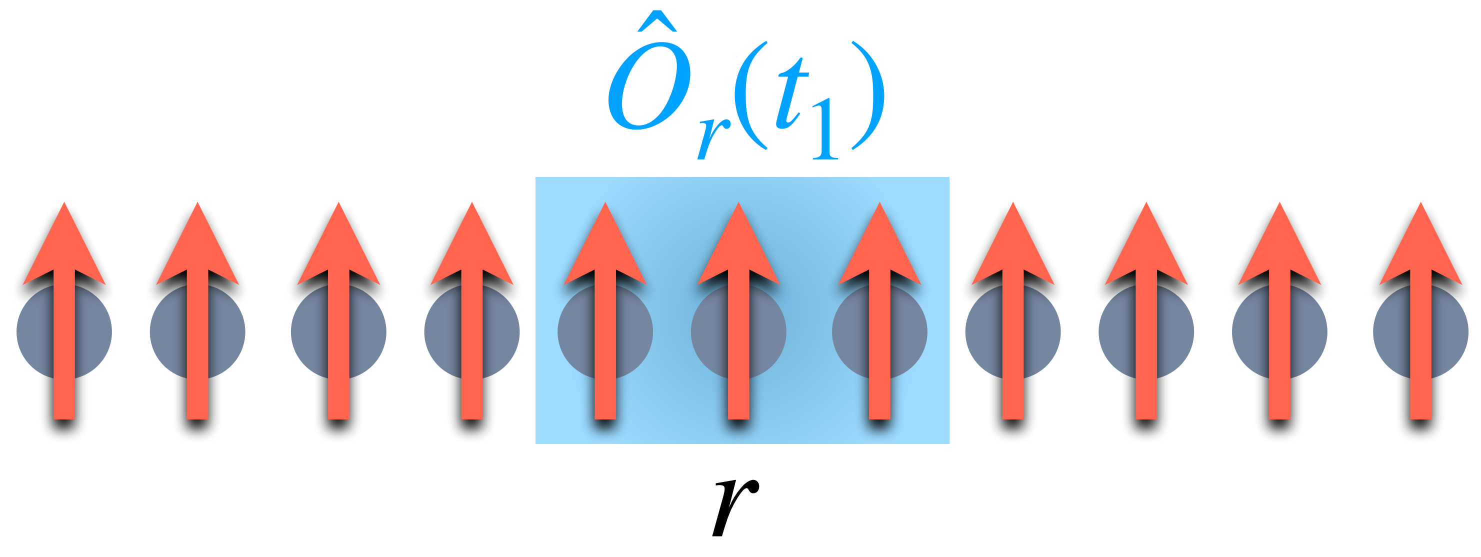
Let $\hat{O} = \hat{O}_r$ be an operator acting at location r (e.g.. $\hat{O}_r = \hat{\sigma}_r^z$ spin operator)

$\hat{O}_r(t)$ acts on an increasingly large region around r



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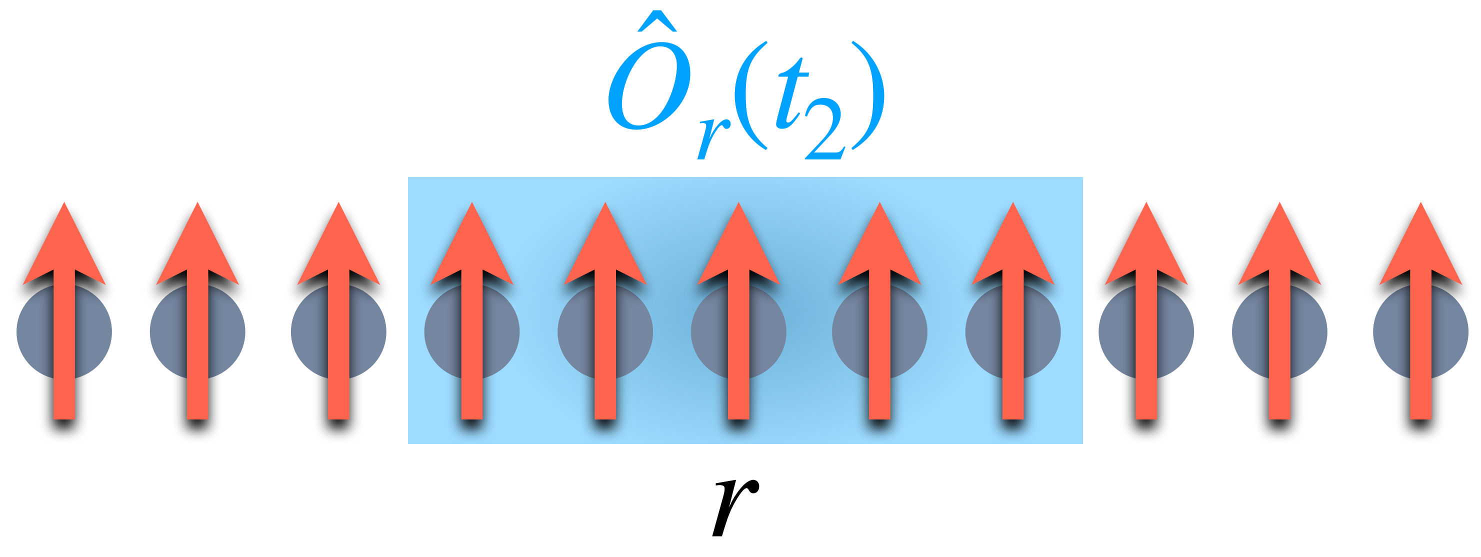

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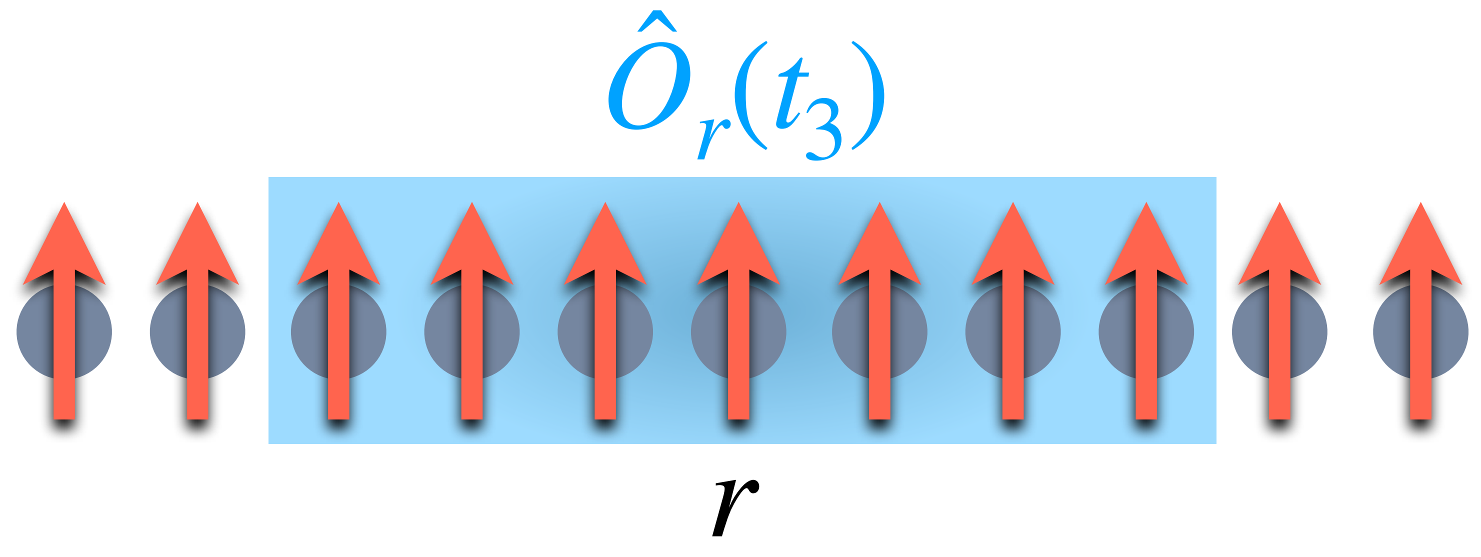

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Schrödinger picture: $|\Psi(t)\rangle = \hat{U}(t)|\Psi\rangle$ ⟶ Heisenberg picture: $\hat{O}(t) = \hat{U}(t)^{\dagger}\hat{O}\hat{U}(t)$

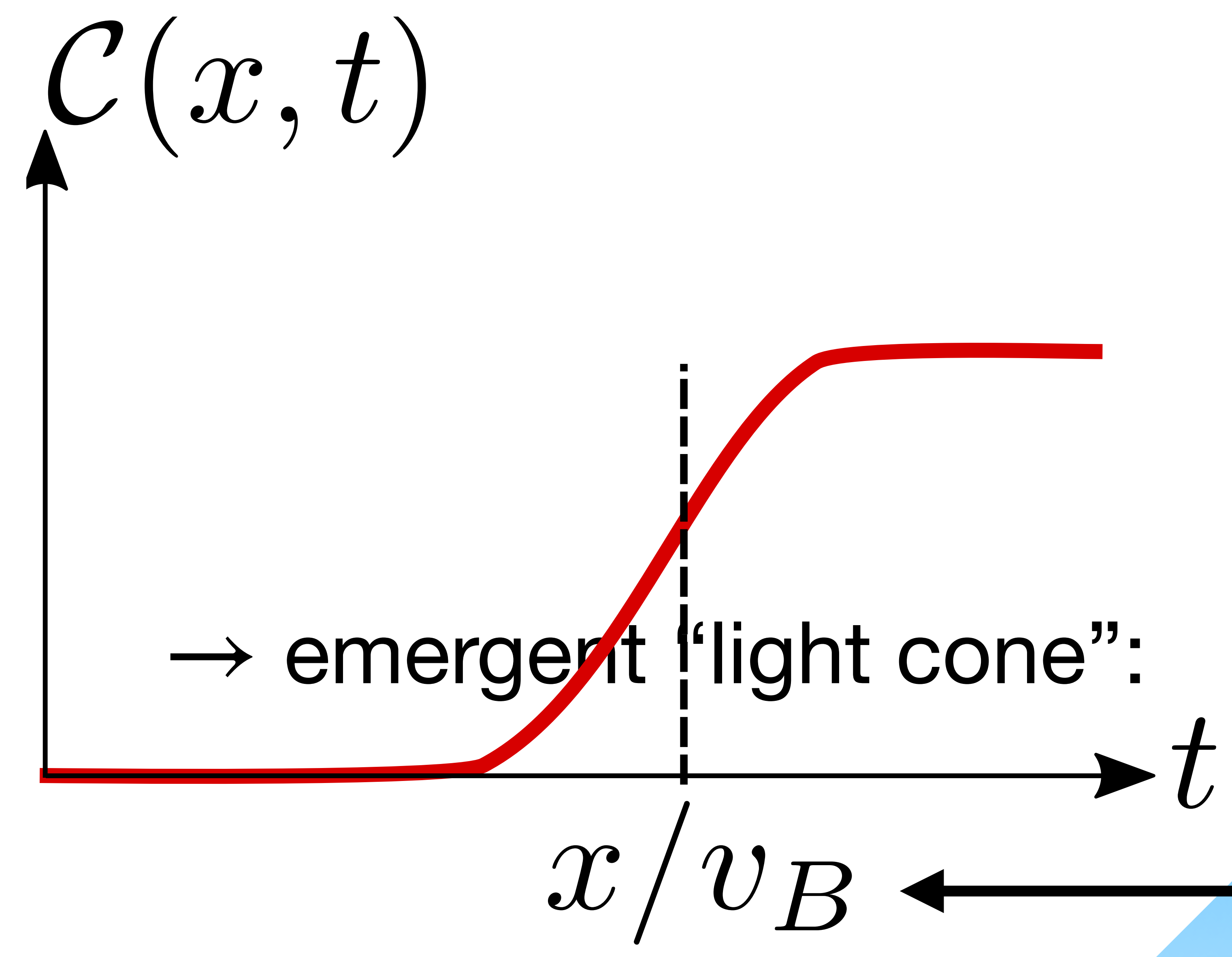
Diagnose using commutator: $[\hat{O}_r(t), \hat{O}'_{r'}]$

Let $\hat{O} = \hat{O}_r$ be an operator acting at location r (e.g. $\hat{O}_r = \hat{\sigma}_r$ spin operator)

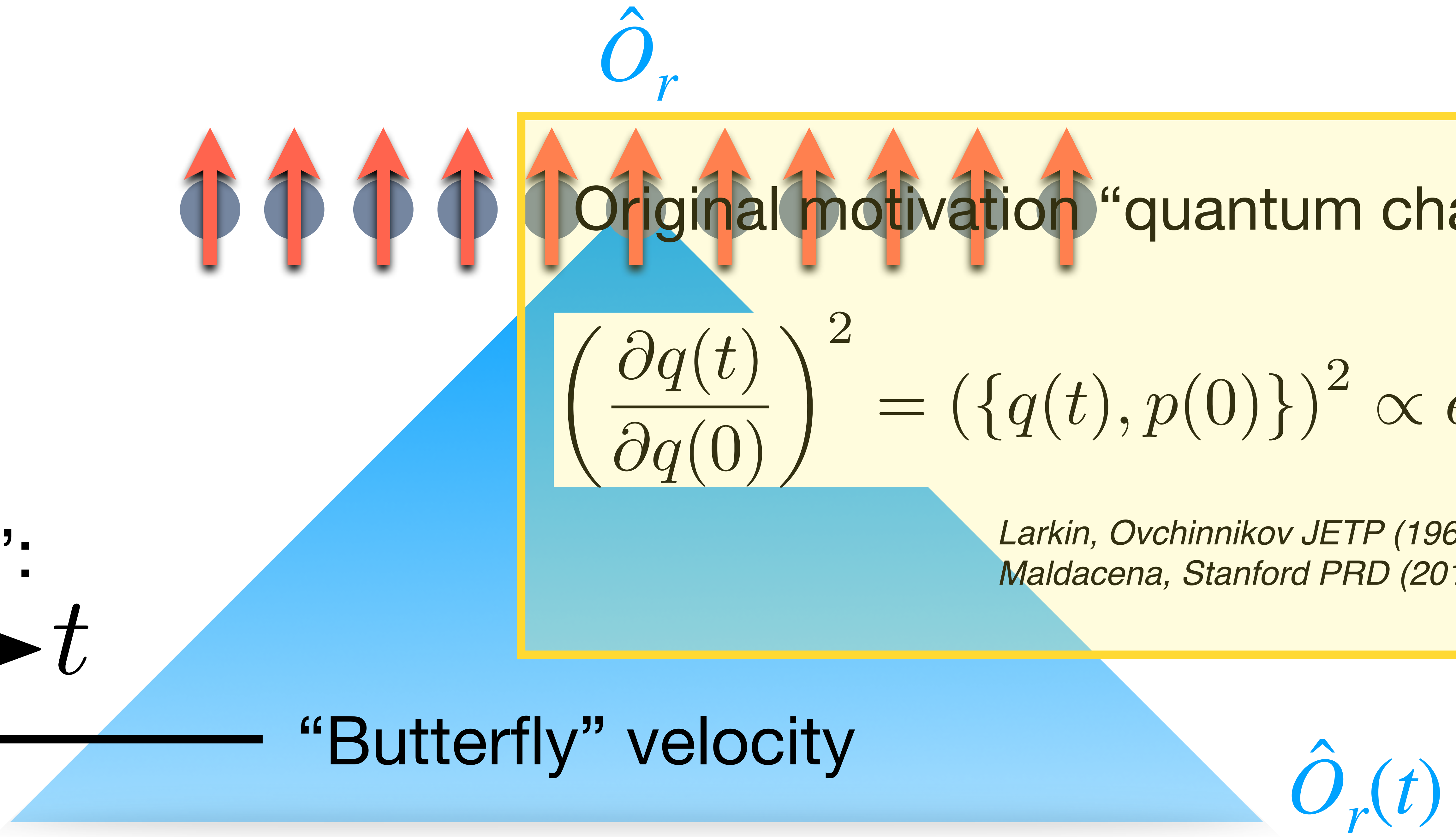
Norm of commutator: $\mathcal{C}(r-r', t) = -\mathrm{Tr}([\hat{O}_r(t), \hat{O}'_{r'}]^2)$

“Out-of-time-order” correlator (OTOC)

$\hat{O}_r(t)$ acts on an increasingly large region around r



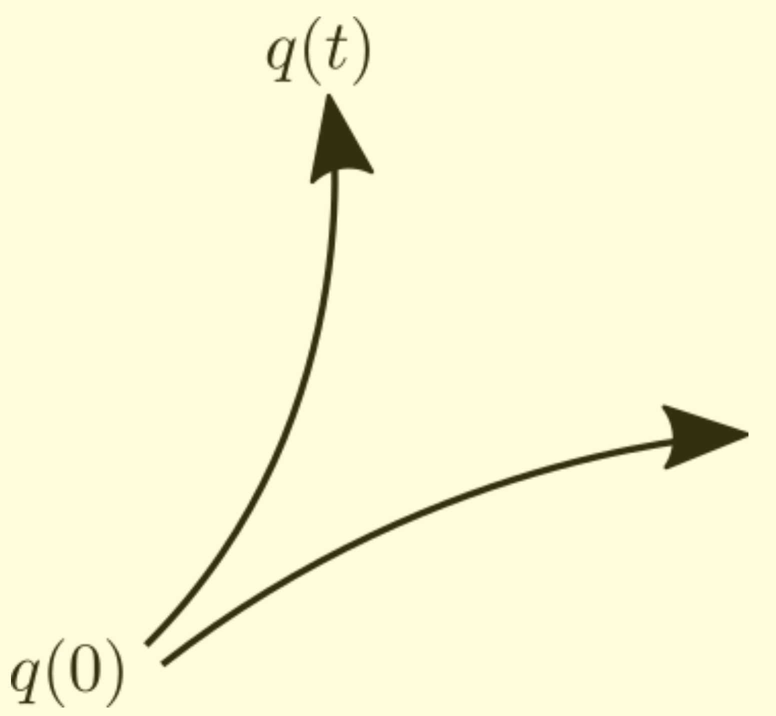
→ emergent “light cone”:



Original motivation “quantum chaos”

$$\left(\frac{\partial q(t)}{\partial q(0)}\right)^2 = (\{q(t), p(0)\})^2 \propto e^{2\lambda_L t}$$

Larkin, Ovchinnikov JETP (1969), Maldacena, Stanford PRD (2016)

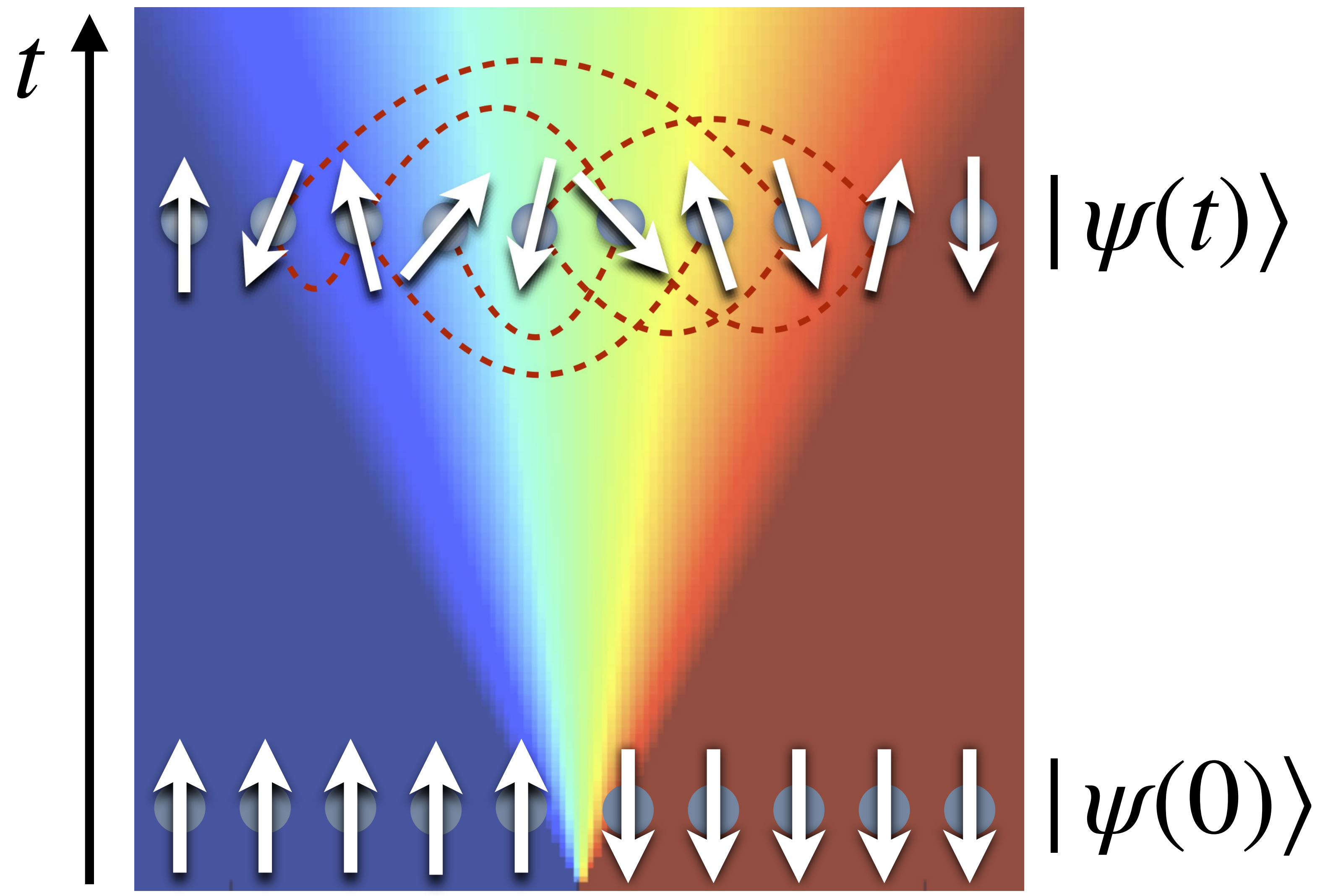


Dynamics is hard to study

Expectation values: $\langle O(t)\rangle = \langle \Psi(t) | O | \Psi(t)\rangle$

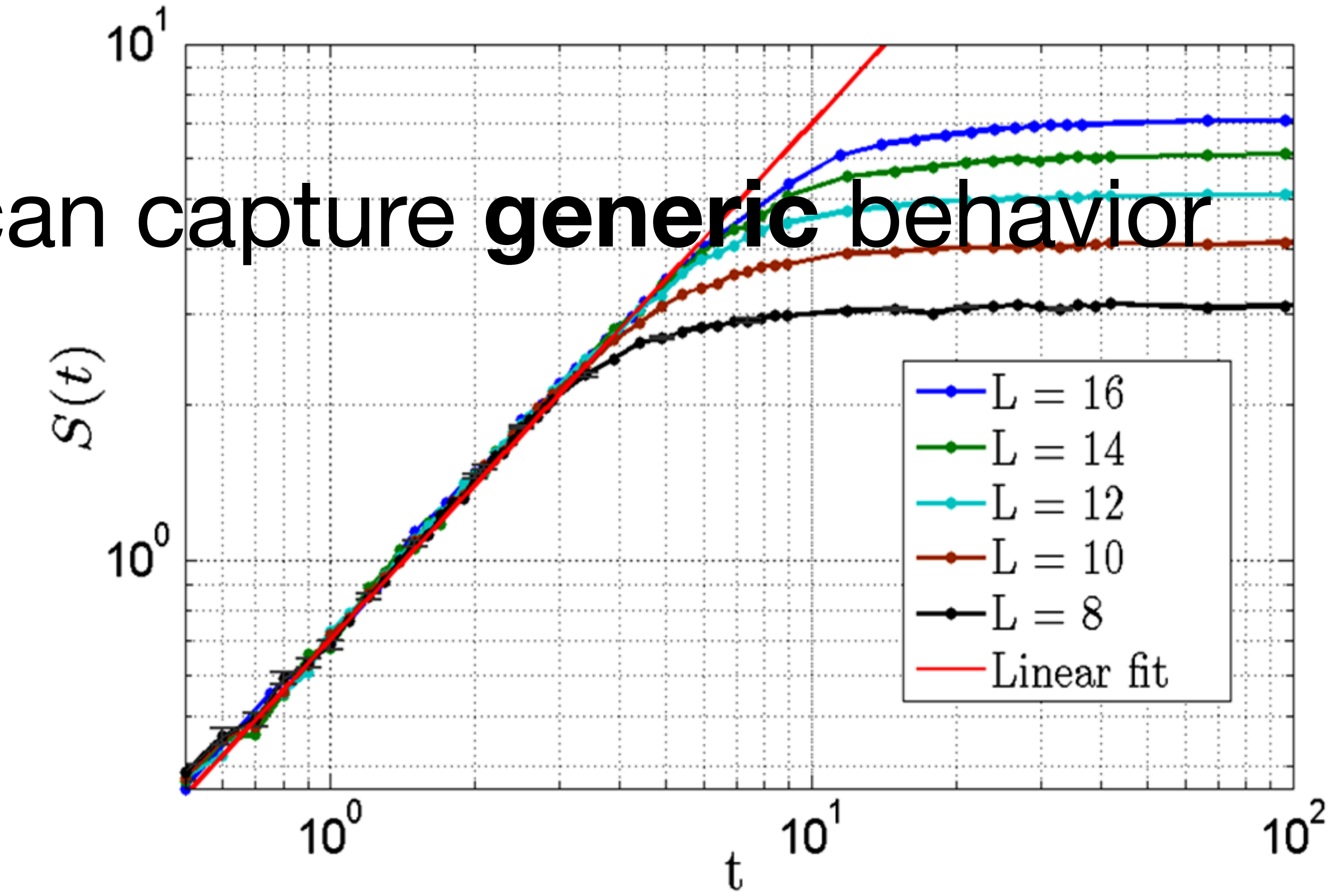
Entanglement entropy: $S_A(t) = -\operatorname{Tr}(\rho_A \ln \rho_A)$

OTOC: $\mathcal{C}(r-r',t) = -\operatorname{Tr}([\hat{O}_r(t), \hat{O}'_{r'}]^2)$



But calculating any of these tends to be **exponentially costly** in the light cone volume $(vt)^D$

⇒ Need **solvable** models that can capture **generic** behavior



Kim, Huse: PRL (2013)

Plan

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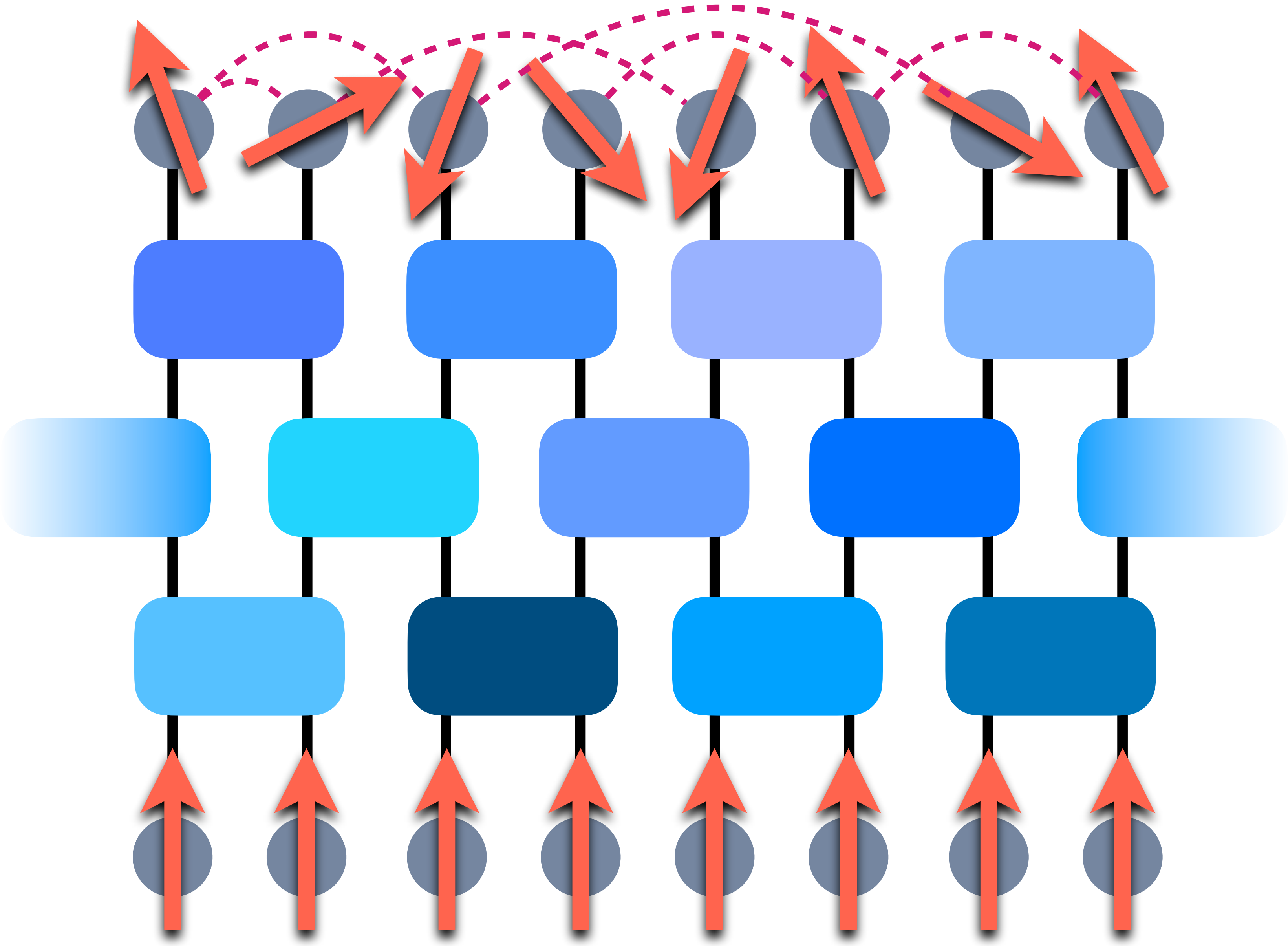
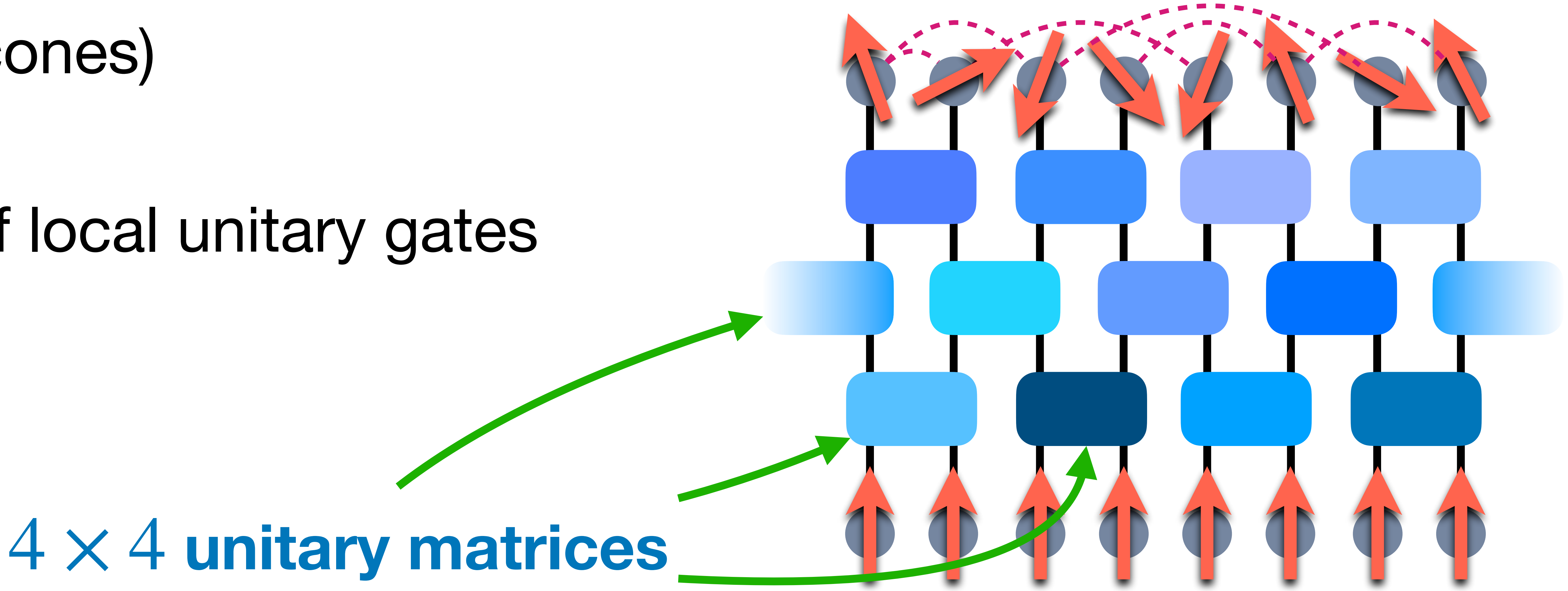


Part 2: Symmetries, measurements and all that

Haar random unitary circuits

We want to keep **unitarity** and **locality** (light cones)

Replace Hamiltonian evolution with a circuit of local unitary gates



To get a solvable model, we need to find appropriate gates

Motivation: Trotter decomposition $e^{-i(H_{\text{even}}+H_{\text{odd}})t} \approx \left(e^{-iH_{\text{even}}\delta t} e^{-iH_{\text{odd}}\delta t}\right)^{t/\delta t}$

Option 1: impose specific structure, e.g. "dual unitarity" *Bertini, Claeys, Prosen: arXiv 25*

Haar measure: uniquely defined by requiring that $P(U) = P(UV) = P(VU)$ for any unitary V

We replace $e^{ih_{i,i+1}\delta t}$ with a more generic unitary ⇒ breaks energy conservation

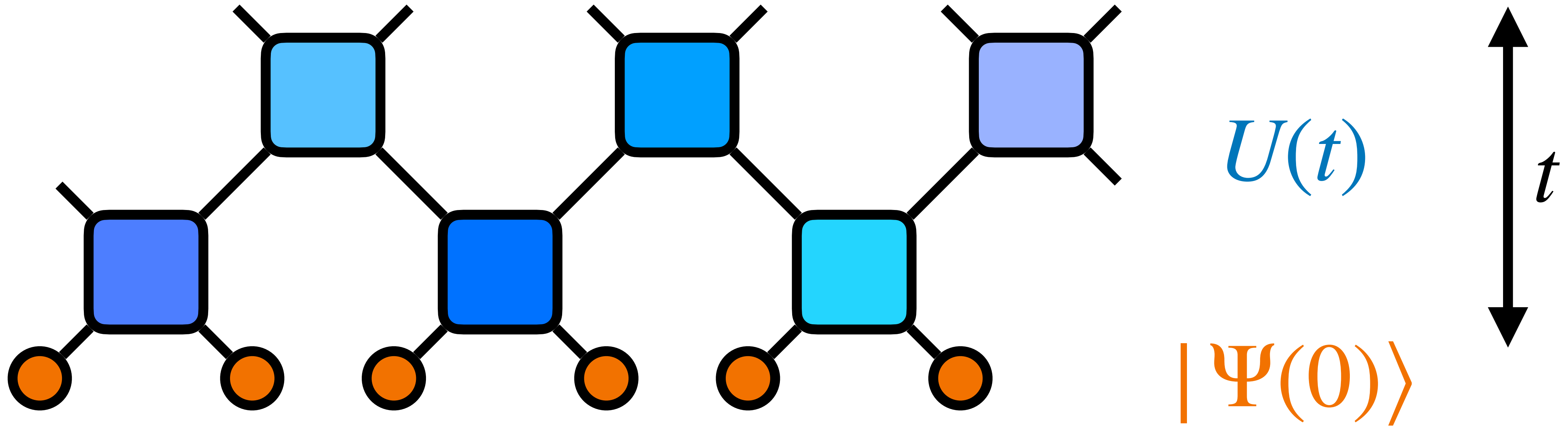
Option 2: make gates **random** and focus on average / typical behavior *Fisher,*

We will choose each gate **independently** from Haar distribution over $U(4)$

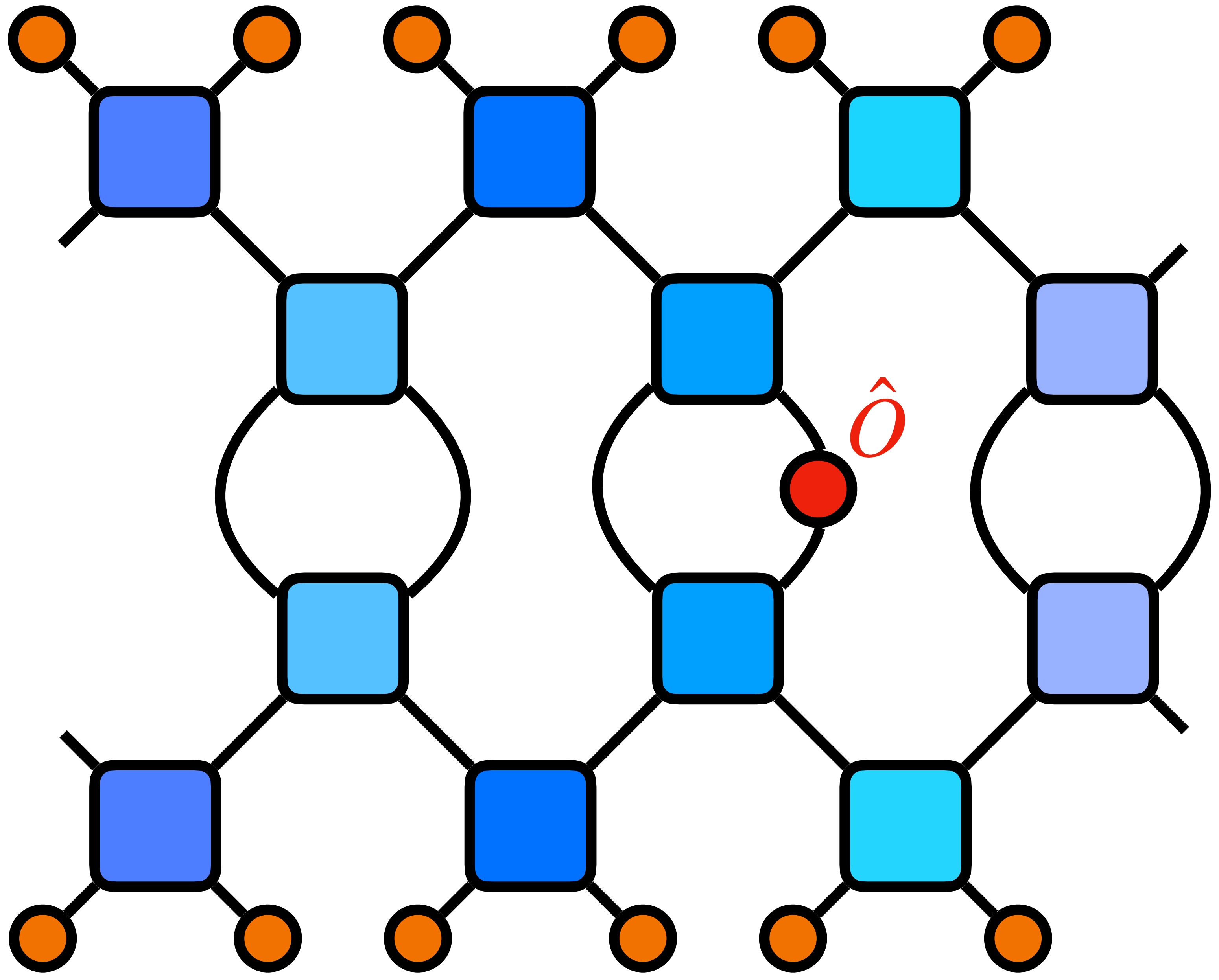
Not this:

Calculating with Haar circuits: a warm-up

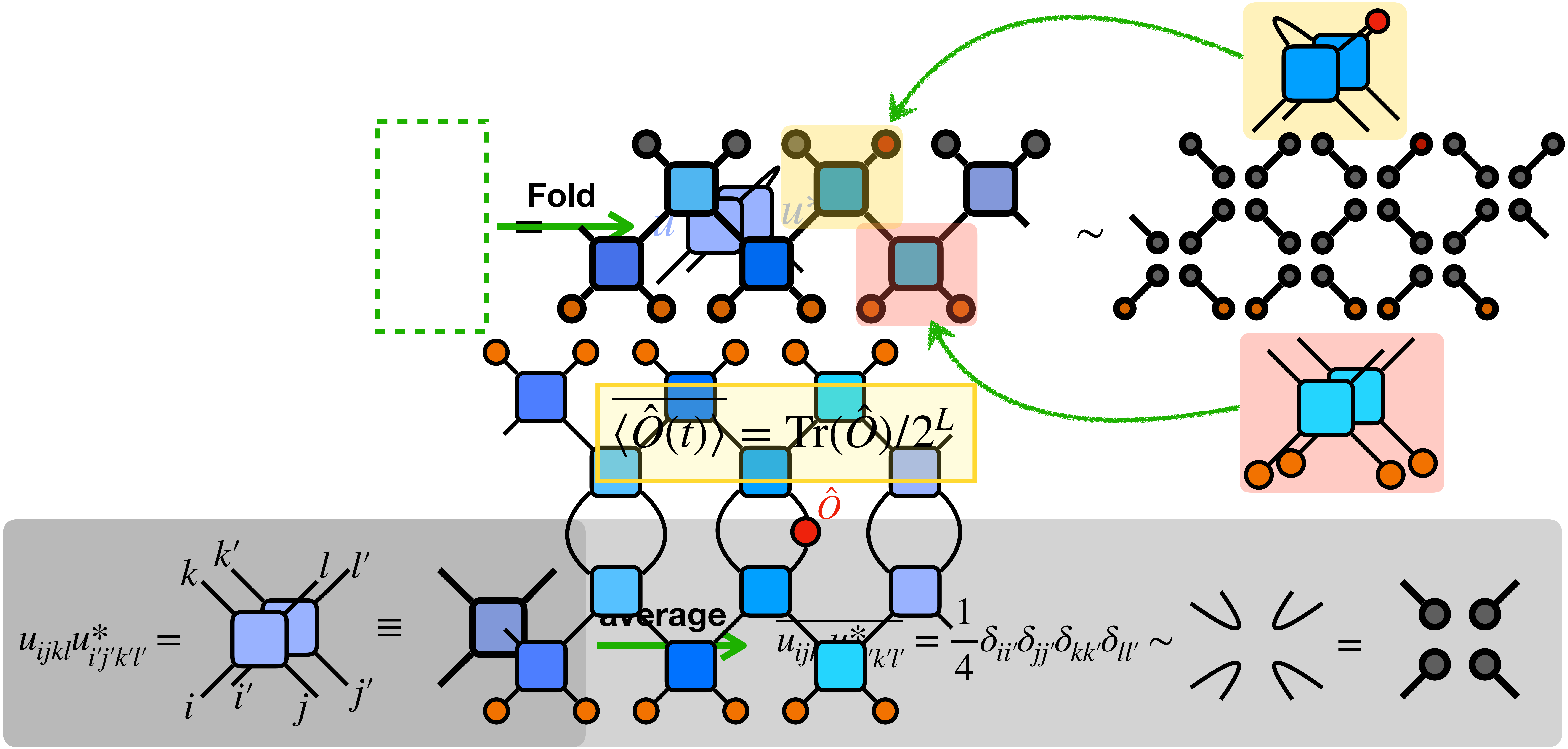
$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle =$



$\langle \hat{O} \rangle(t) = \langle \Psi(t)|\hat{O}|\Psi(t)\rangle =$



Calculating with Haar circuits: a warm-up



$$\overline{\langle \hat{O}(t) \rangle} = \mathrm{Tr}(\hat{O})/2^L$$

$$u_{ijkl}u^*_{i'j'k'l'} = \equiv$$

$$\overline{u_{ijkl}u^*_{i'j'k'l'}} = \frac{1}{4}\delta_{ii'}\delta_{jj'}\delta_{kk'}\delta_{ll'} \sim \quad =$$

Let's do something more interesting: entanglement

Von Neumann entropy: $S_{\mathrm{vN}}(t) = -\,\mathrm{Tr}(\rho_A \ln \rho_A)$ ⟵ Involves all powers of ρ

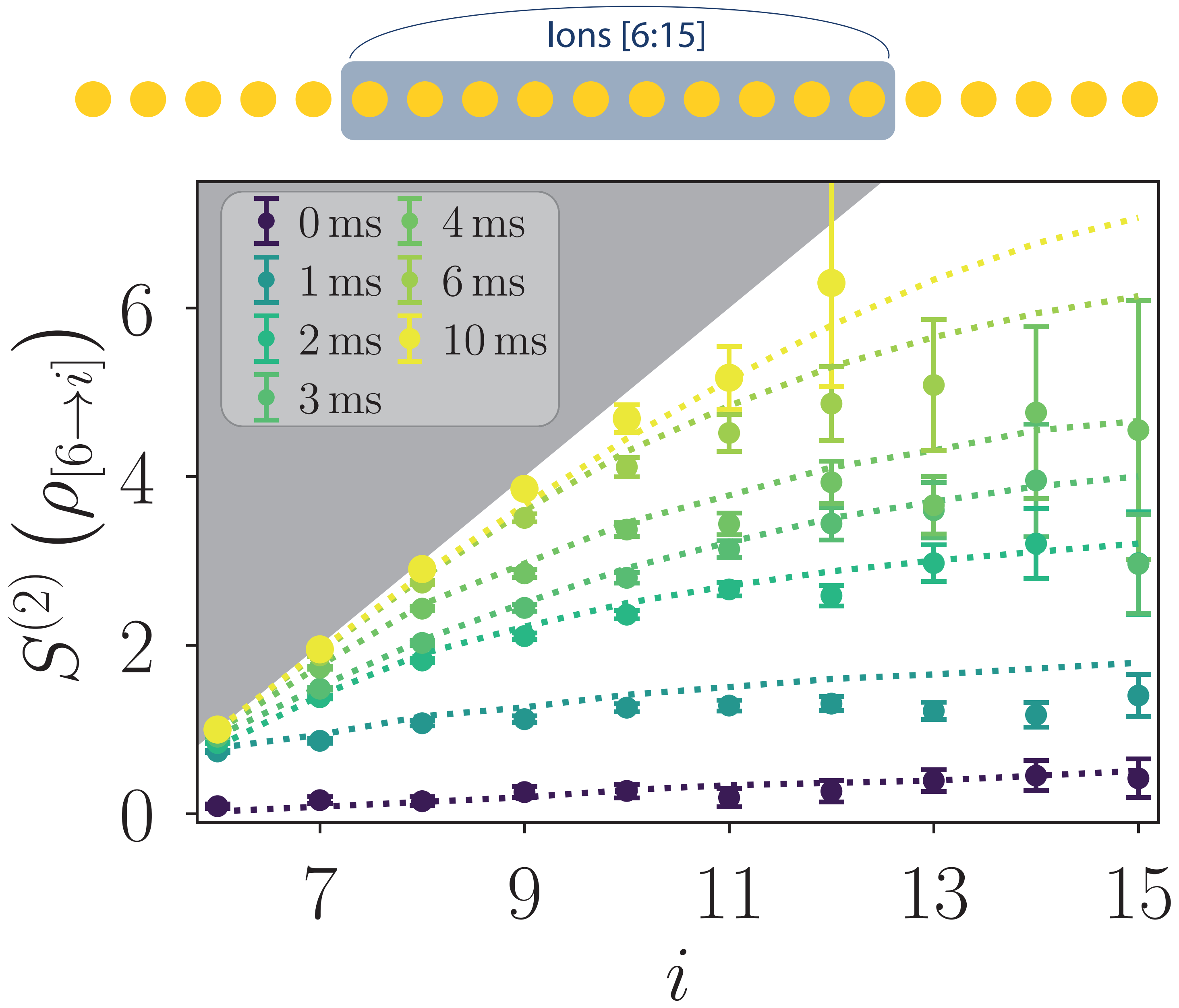
Rényi entropies: $S_n(t) = \dfrac{1}{1-n} \ln \mathrm{Tr}(\rho_A^n)$ ⟵ often easier to calculate

$$S_{\mathrm{vN}} = \lim_{n\to 1} S_n$$

Also to measure!

We will calculate $S_2^{\mathrm{ann.}} = -\log \overline{\mathrm{Tr}(\rho_A^2)}$

"Annealed" average of 2nd Rényi entropy

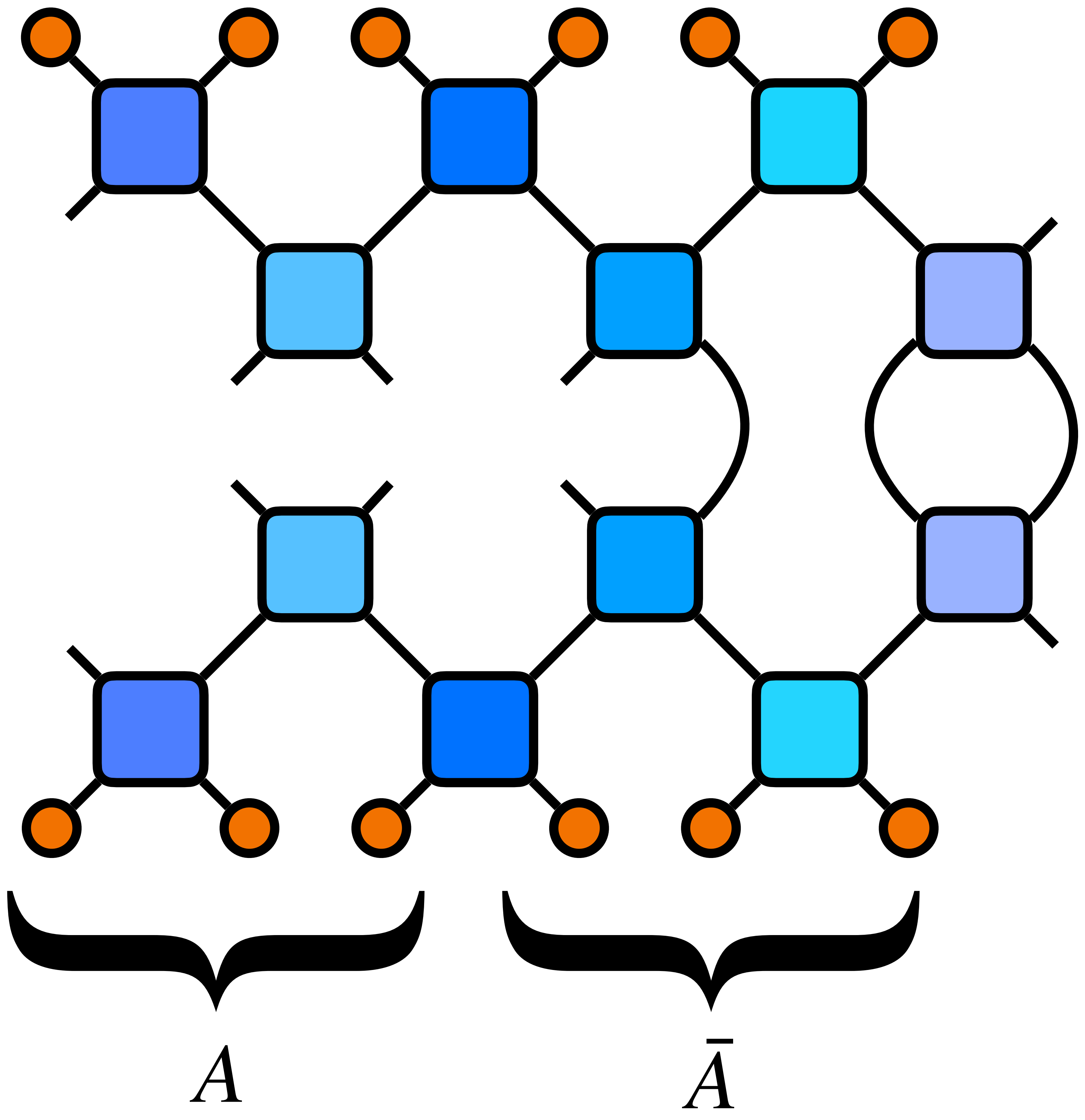
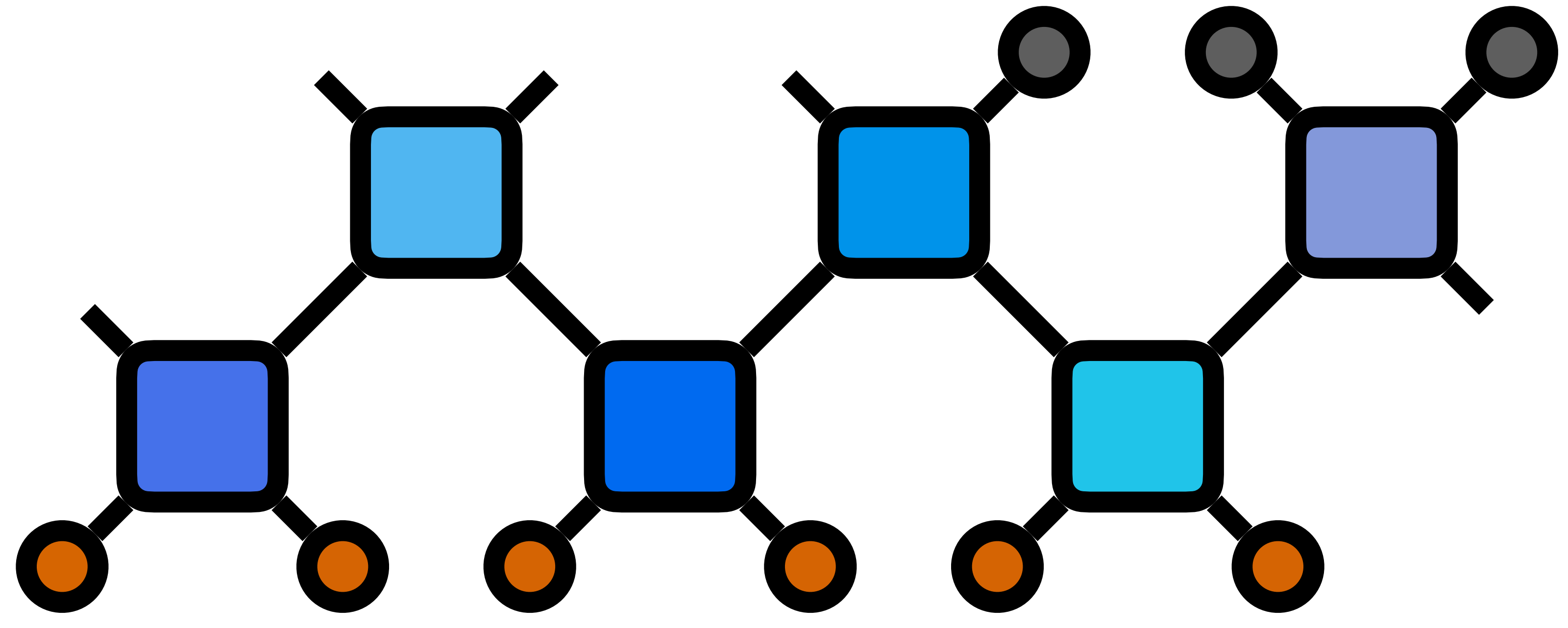


Brydges et al., Science (2019)

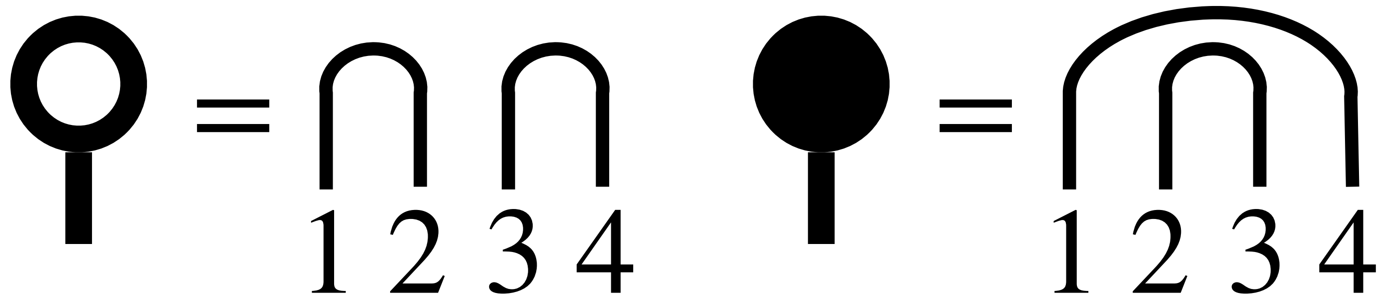
Kaufman et al., Science (2016)

Islam et al., Nature (2015)

Calculating $\overline{\mathrm{Tr}(\rho_A^2)}$

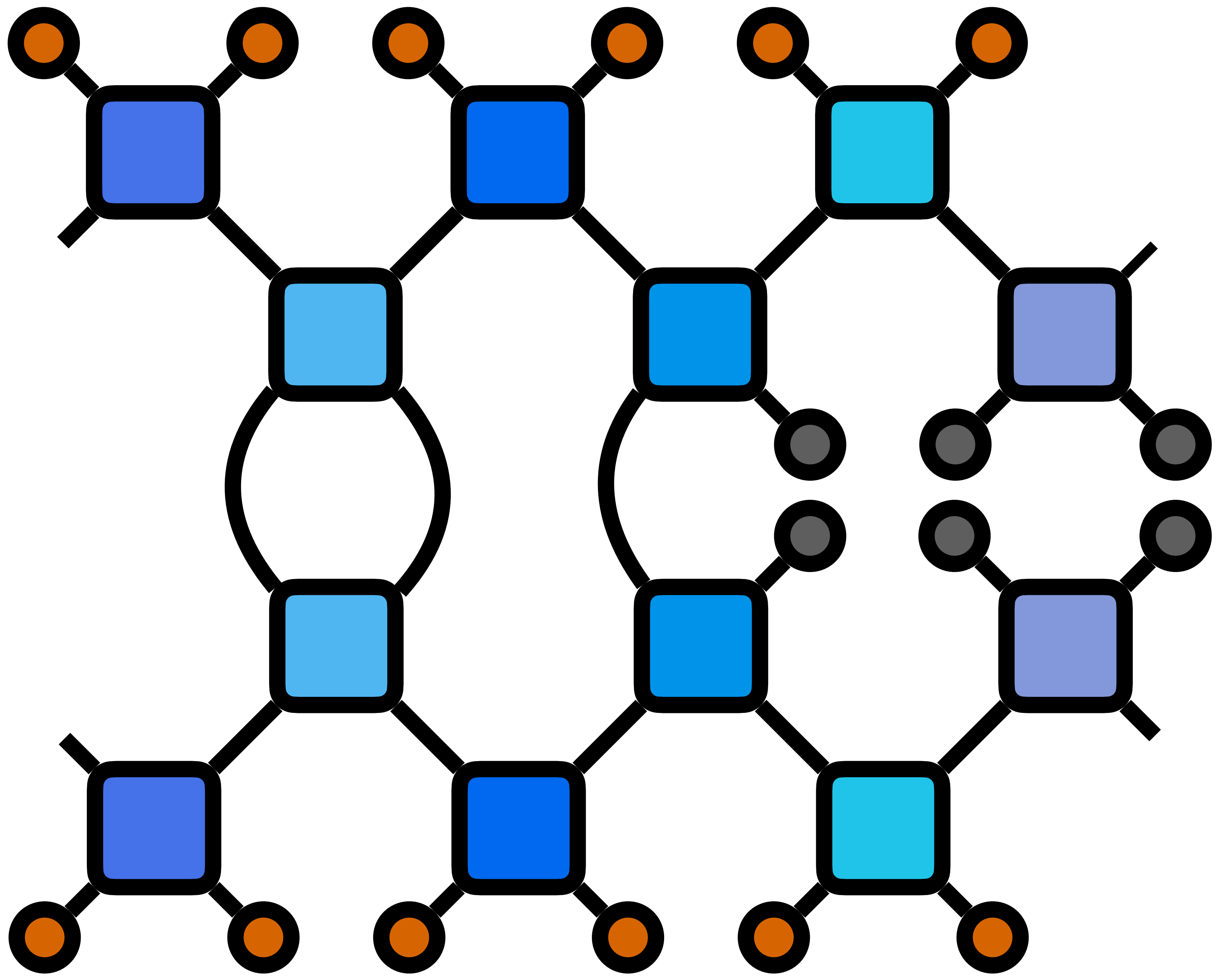
$\rho_A(t) = \mathrm{Tr}_{\bar{A}}(|\Psi(t)\rangle\rangle\langle\Psi(t)|) =$  $=$ 

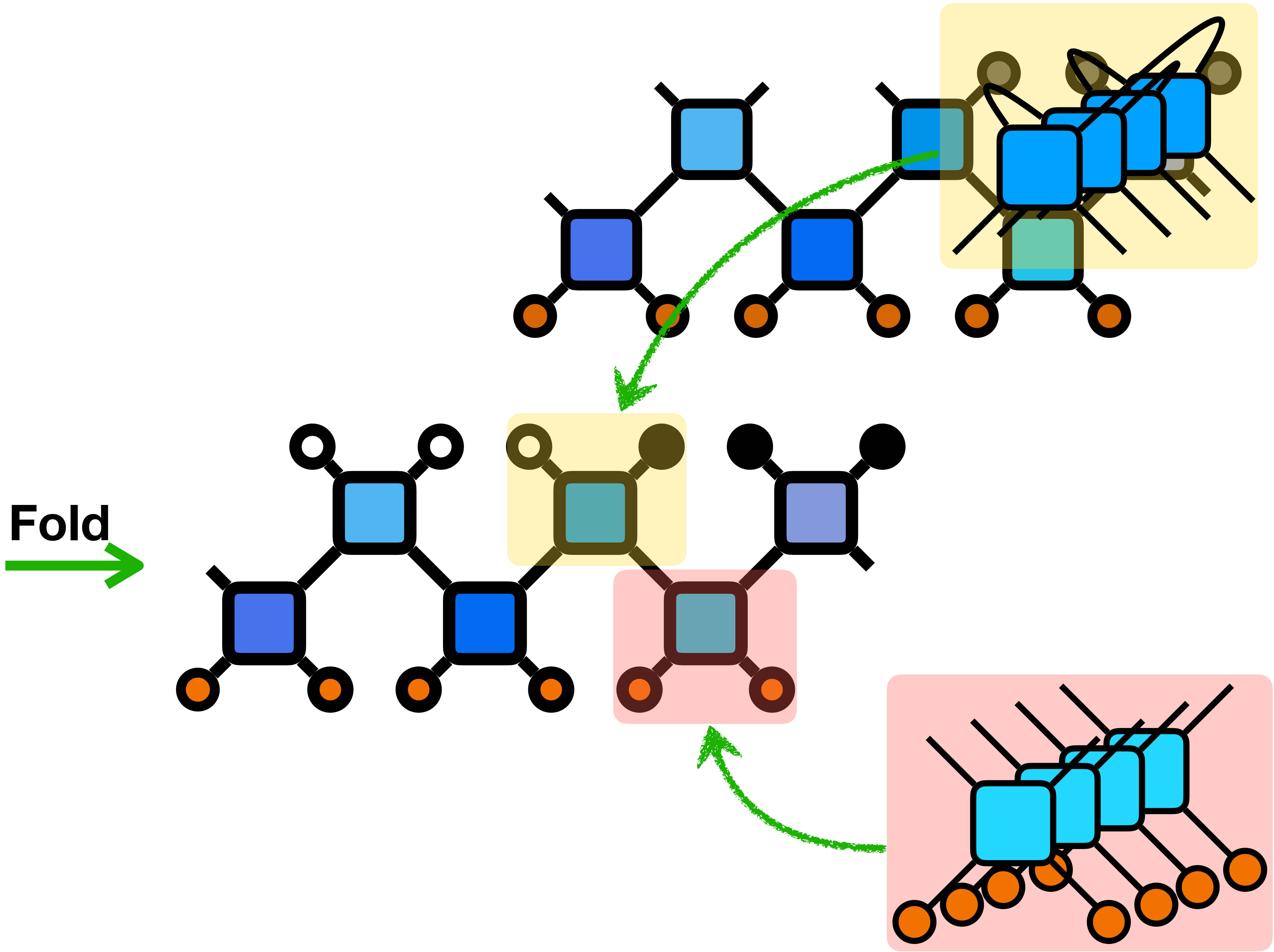
Calculating $\overline{\mathrm{Tr}(\rho_A^2)}$



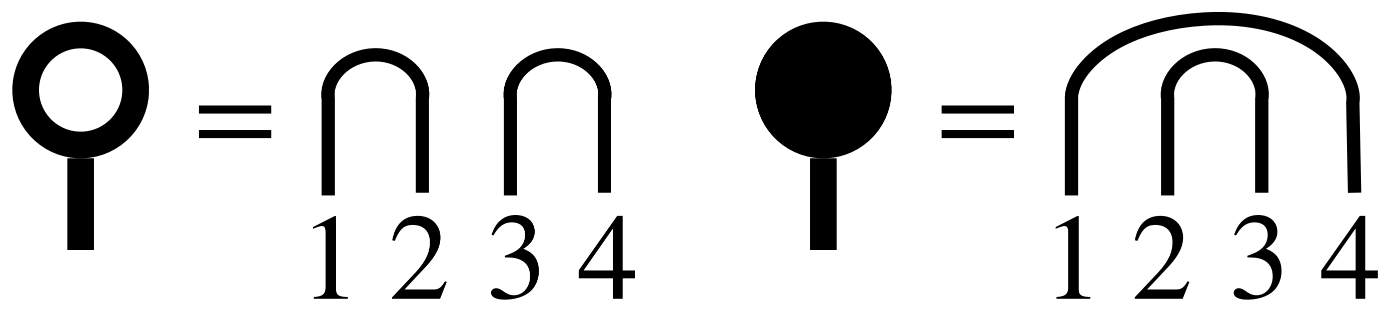
$$\rho_A(t) = \mathrm{Tr}_{\bar{A}}(\,|\,\Psi(t)\rangle\rangle\langle\Psi(t)\,|\,) =$$

$$\mathrm{Tr}(\rho_A(t)^2) =$$

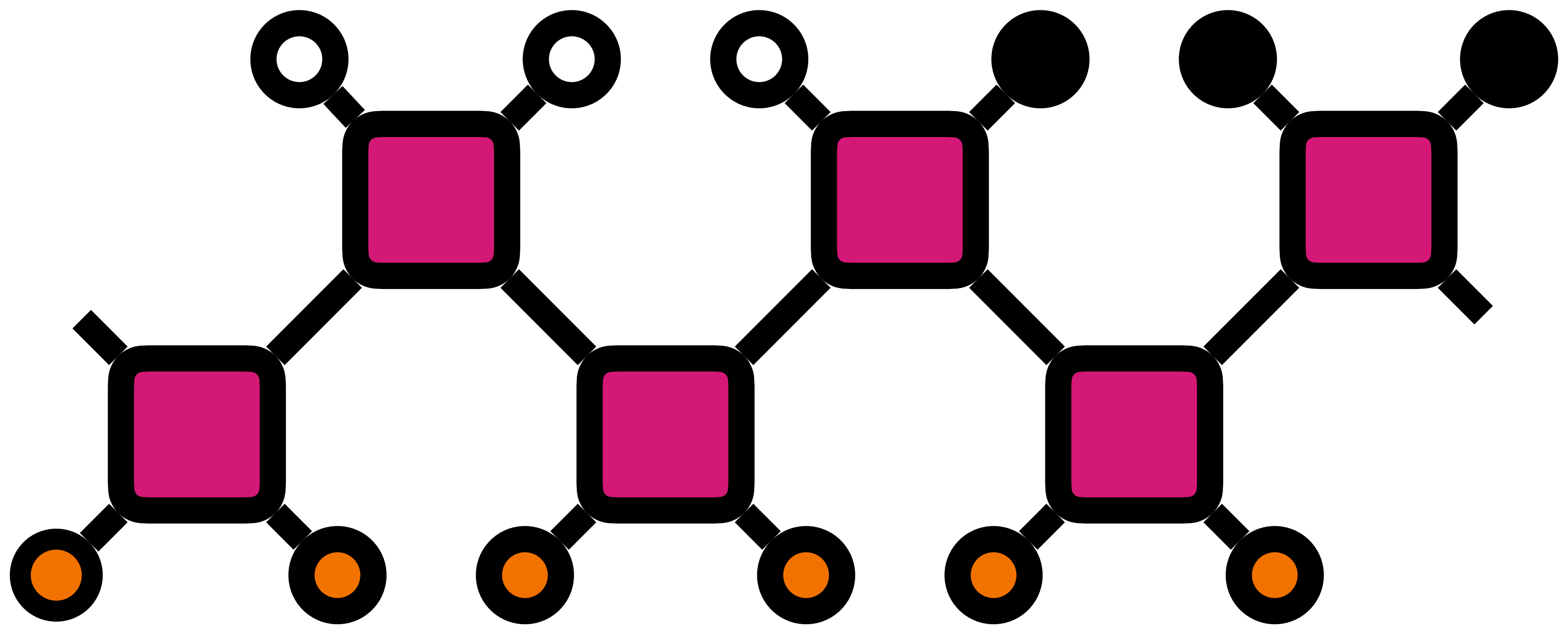




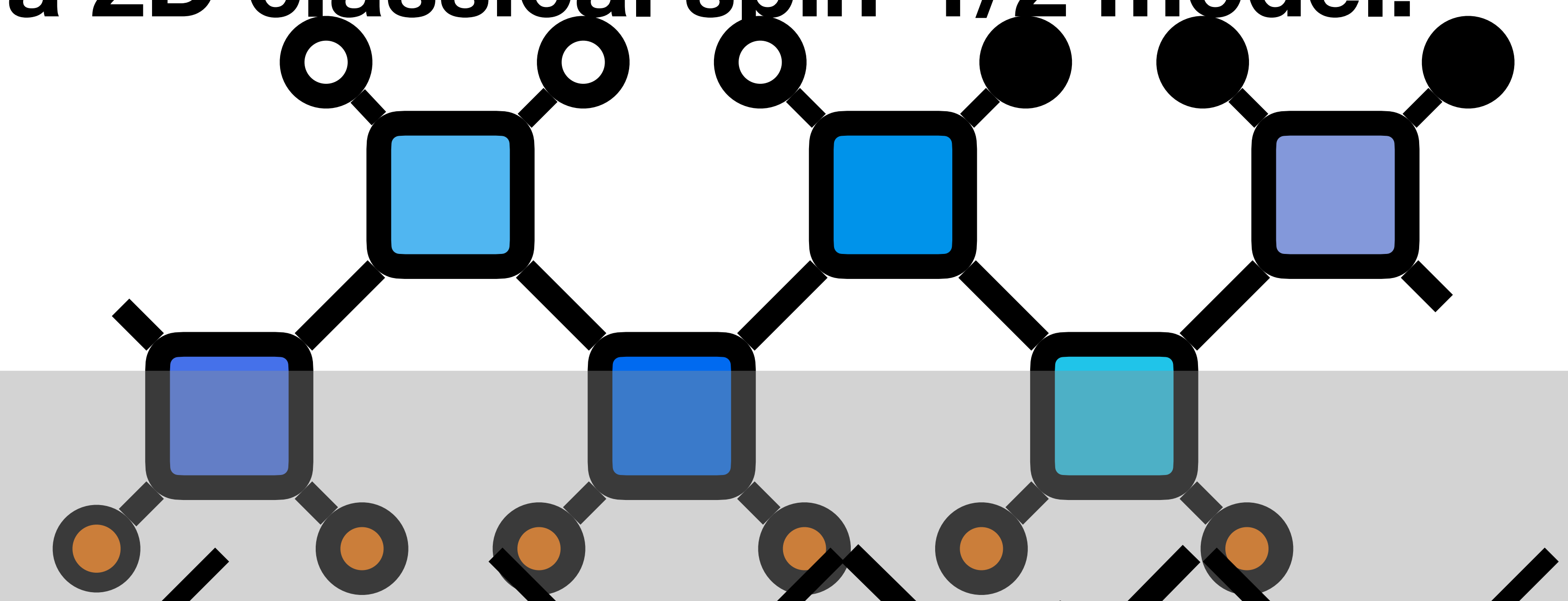
Calculating $\overline{\mathrm{Tr}(\rho_A^2)}$



$\overline{\mathrm{Tr}(\rho_A(t)^2)} =$

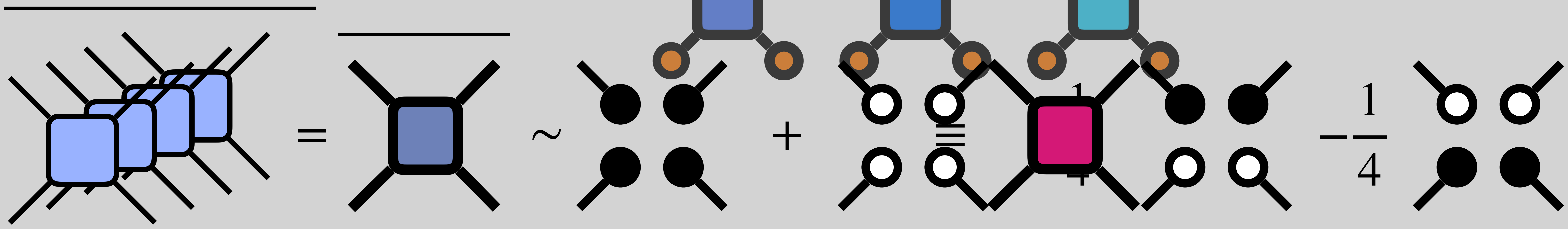


We mapped the average to the partition function of a 2D classical spin-1/2 model!



$\mathrm{Tr}(\rho_A(t)^2) =$

$\overline{u \otimes u^* \otimes u \otimes u^*} =$



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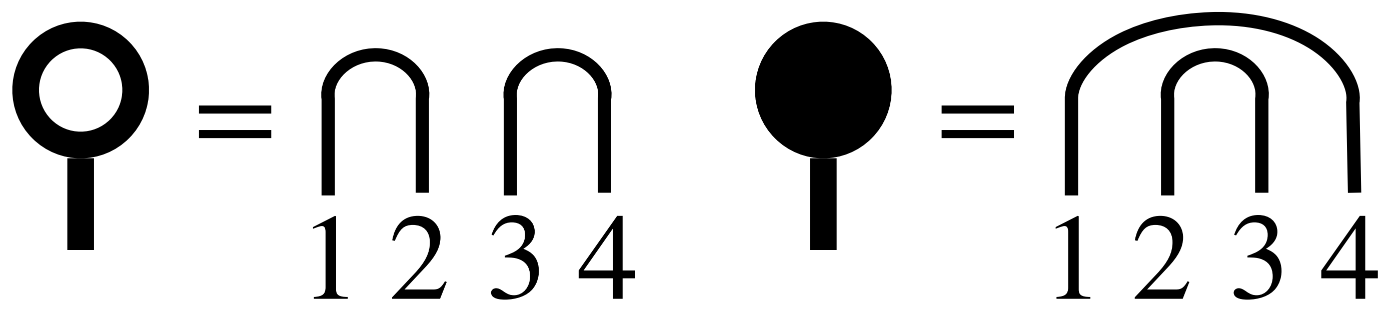

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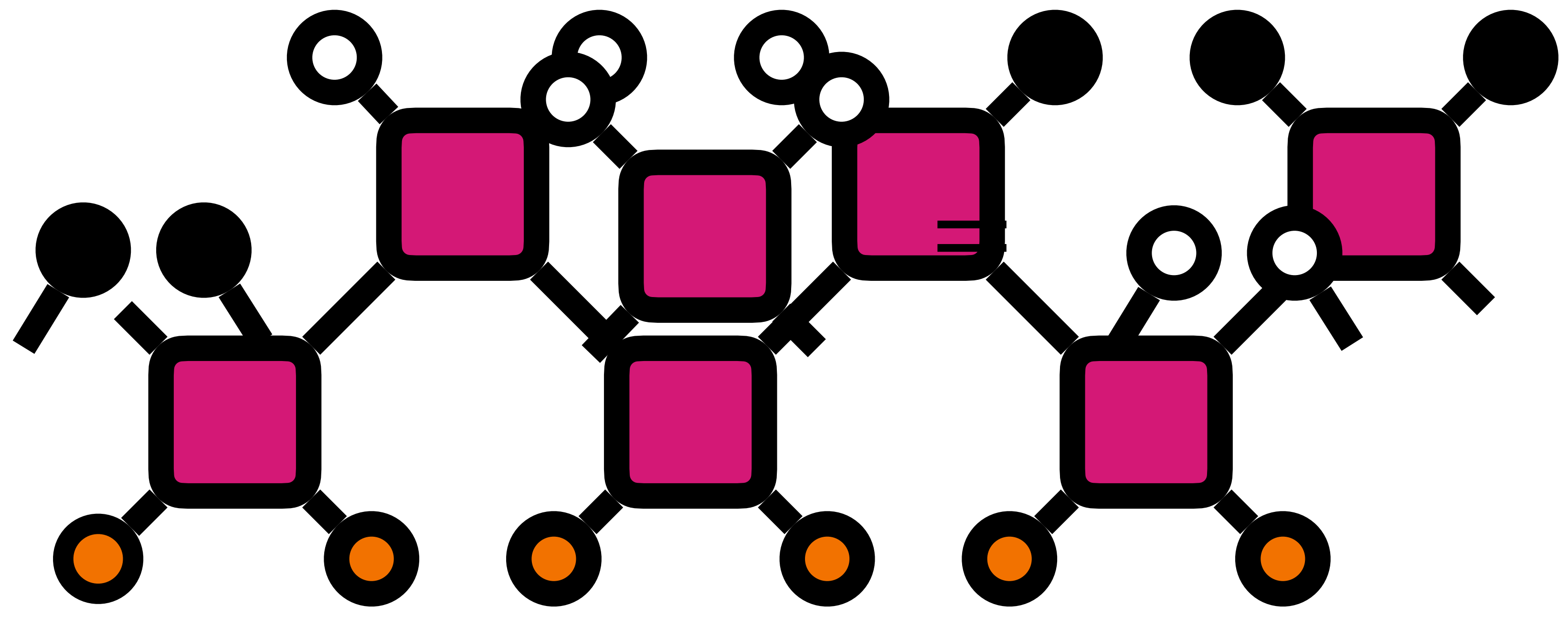
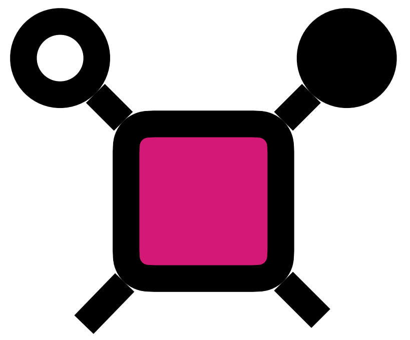
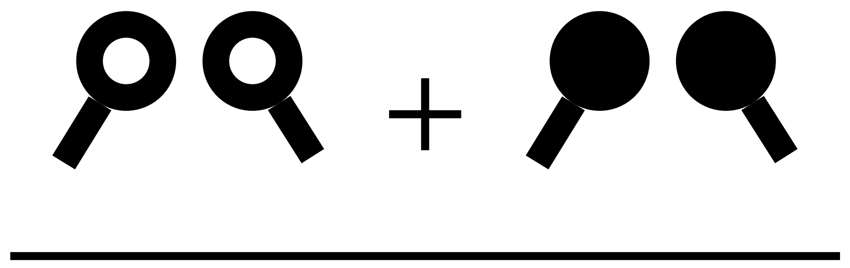


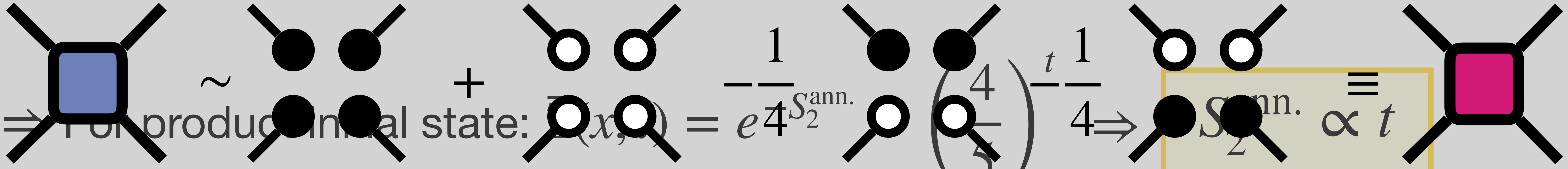


$= \frac{4}{5}$



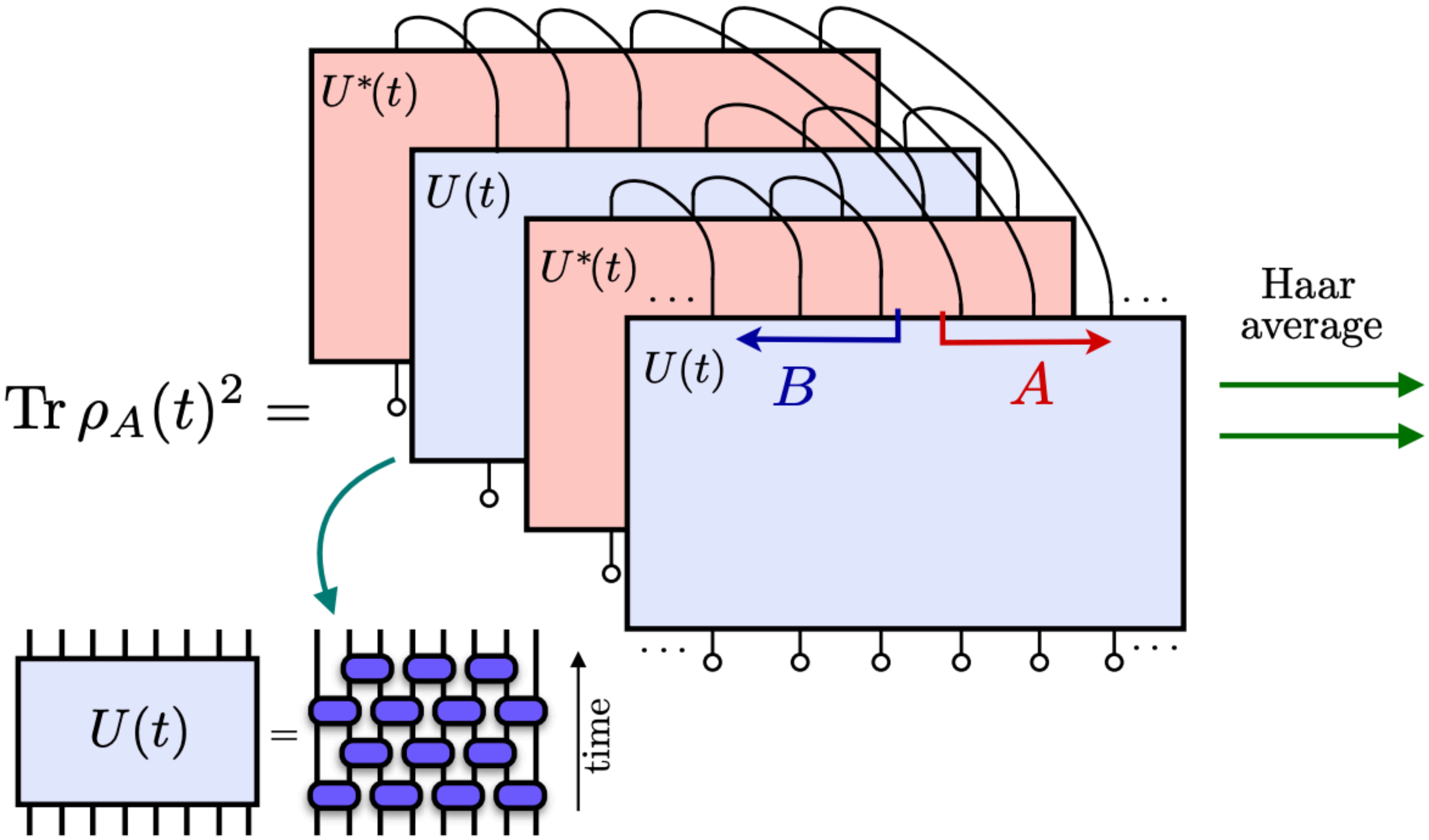
Domain wall random walk

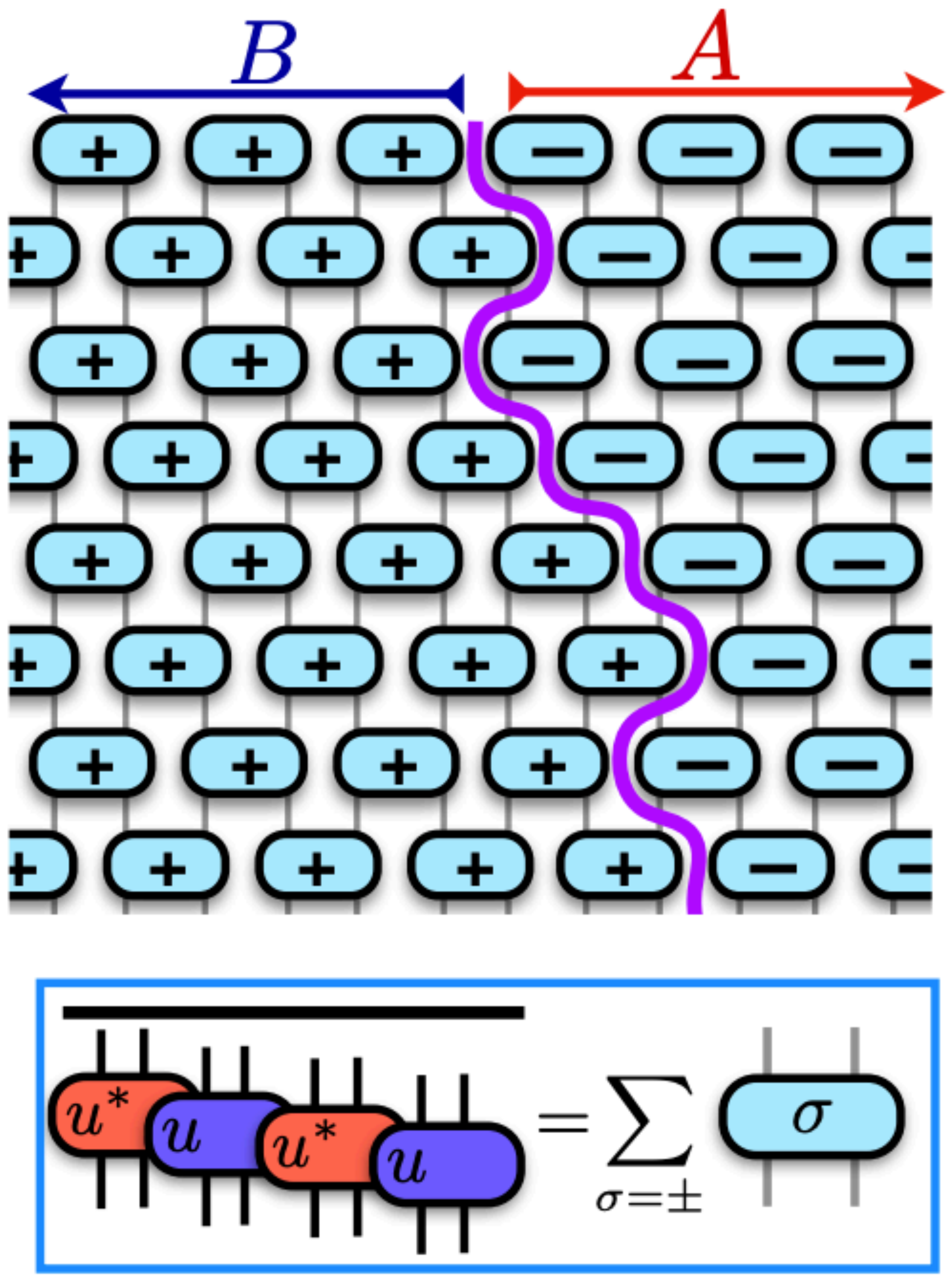
$\Rightarrow$ Recursion: $Z(x,t) := \mathrm{Tr}(\rho_{[0,x]}(t)^2)$ obeys $\overline{Z}(x,t) = \frac{4}{5}\frac{\overline{Z}(x-1,t-1)+\overline{Z}(x+1,t-1)}{2}$



$\Rightarrow$ For product initial state: $\overline{Z}(x,0) = e^{-\frac{1}{4}S_2^{\mathrm{ann.}}} \left(\frac{4}{5}\right)^{t} - \frac{1}{4} \Rightarrow S_2^{\mathrm{ann.}} \propto t$

Statistical mechanics of entanglement growth





Fisher, Khemani, Nahum, Vijay (2023)

$\overline{e^{-S_2}}$ $\leftrightarrow$ domain wall free energy

Growth rate of entropy $\leftrightarrow$ line tension (ferromagnet)

Generalizes to non-random dynamics

Jonay, Huse, Nahum: arXiv 1803.00089

Zhou, Nahum: PRX (2020)

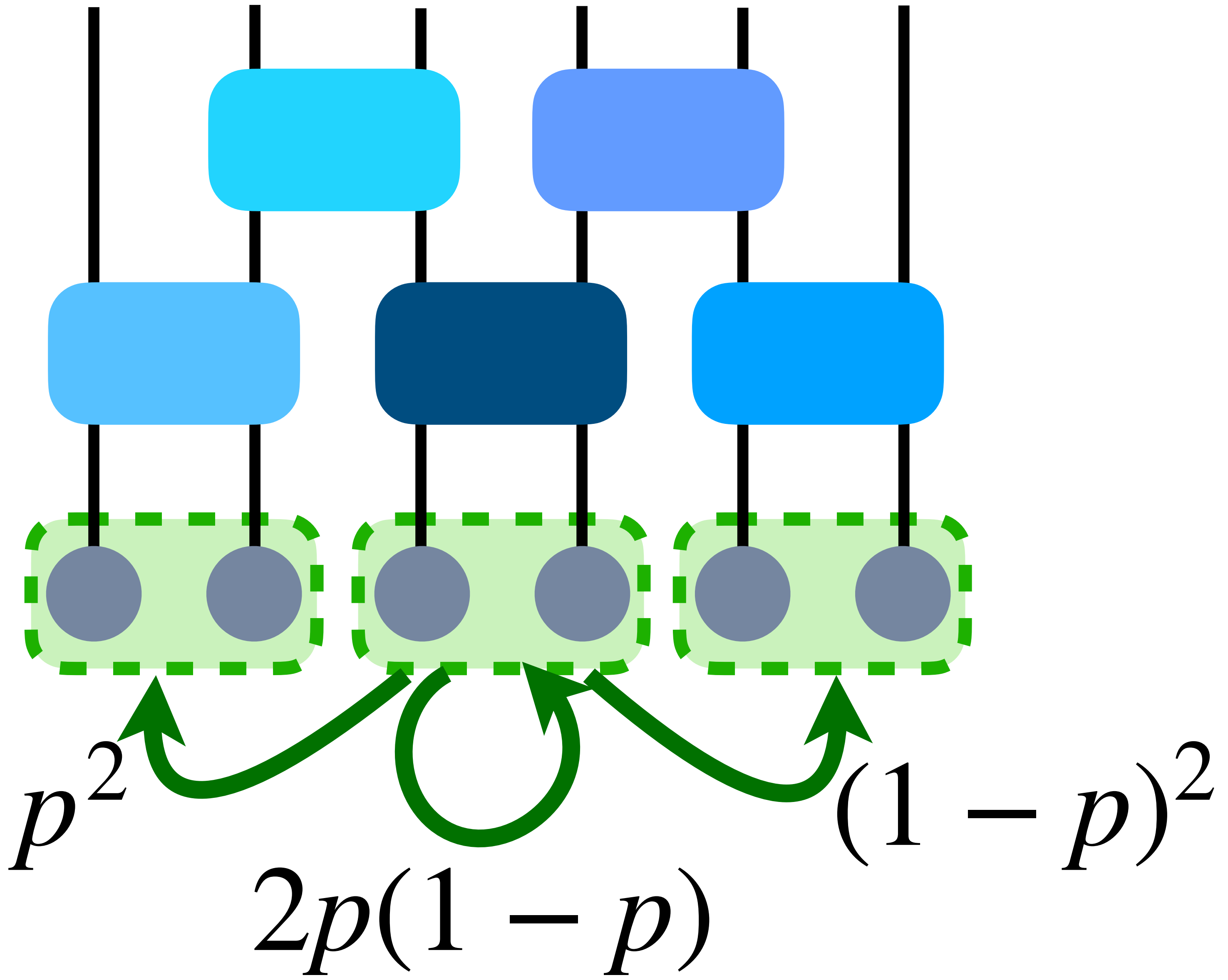
“Hydrodynamics” of operator growth

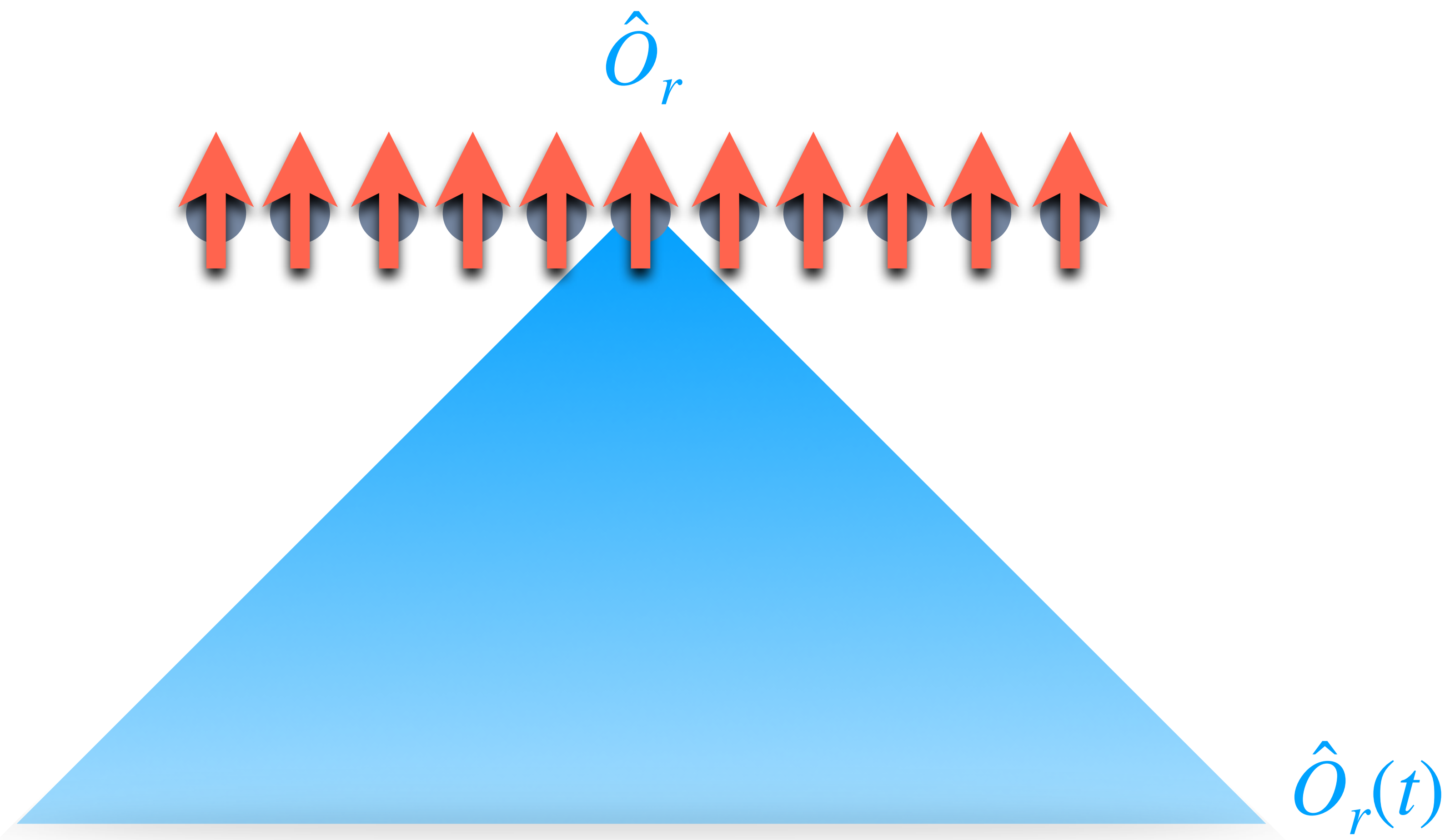
OTOC: $\mathcal{C}(r-r',t) = -\mathrm{Tr}([\hat{O}_r(t), \hat{O}'_{r'}]^2)$ ⟵ Also involves 4 copies of U

$\hat{O}(t) = U^\dagger(t)\hat{O}U(t)$

$\rho_R(x,t)$: probability that operator spread to distance x by time t If $x>0$: $\rho_R(x,t) \approx \partial_x \mathcal{C}(x,t)$

Obeys an **exact** random walk equation





Von Keyserlingk, TR, Pollmann, Sondhi: PRX (2018)

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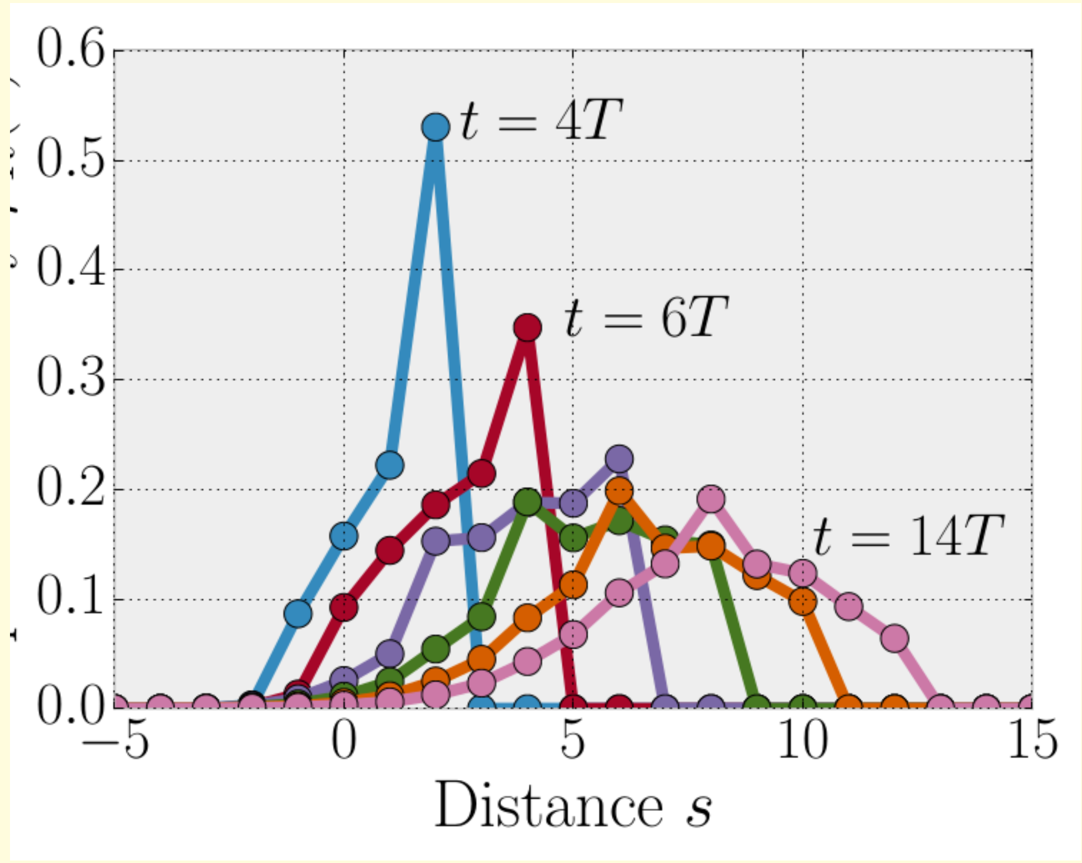
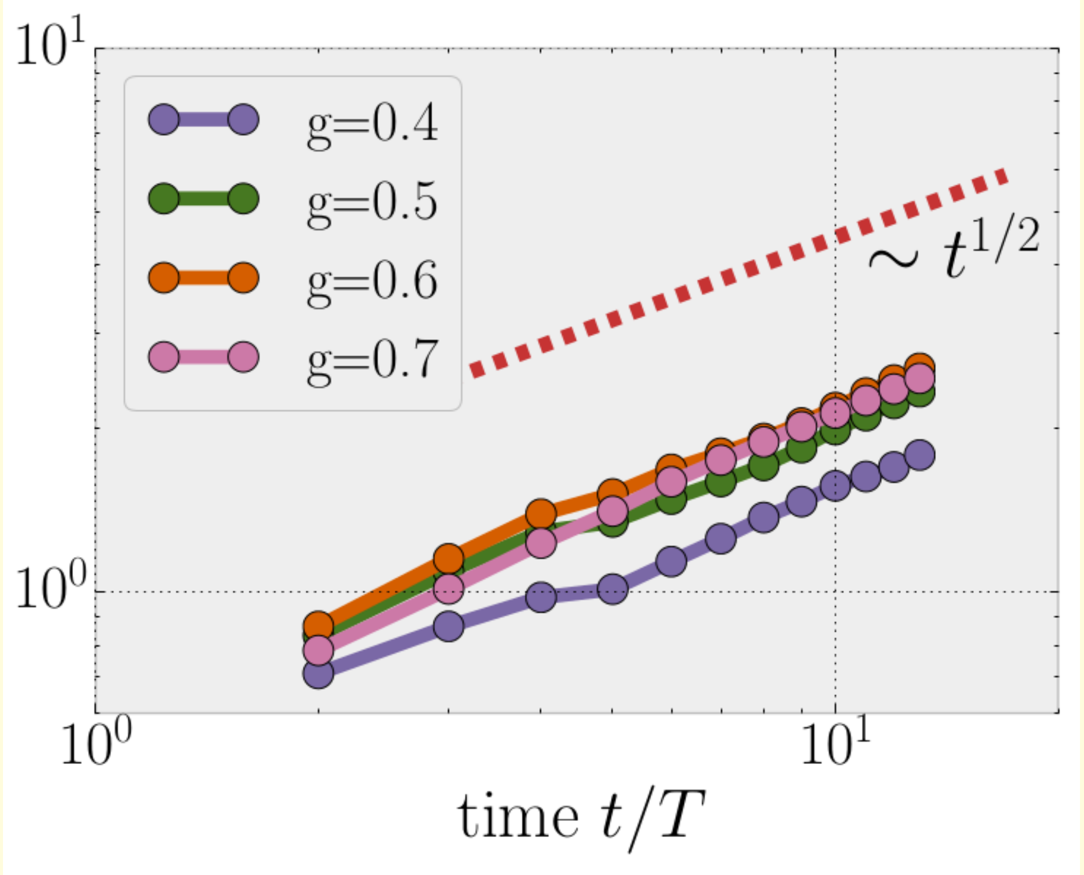
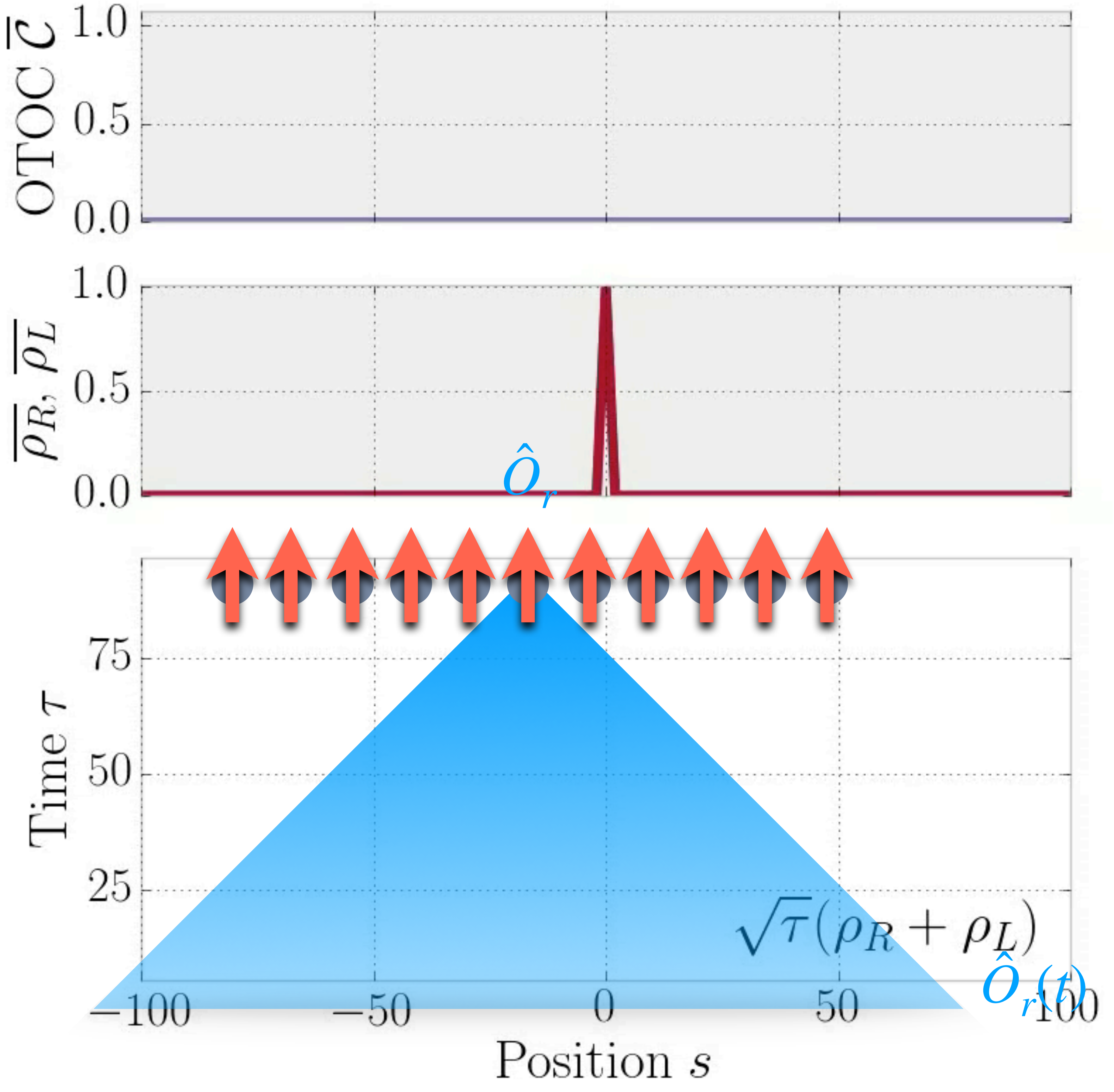
⇒ **biased diffusion** $\partial_t \rho_R = v_B \partial_x \rho_R + D \partial_x^2 \rho_R$

Similar behavior in deterministic systems:



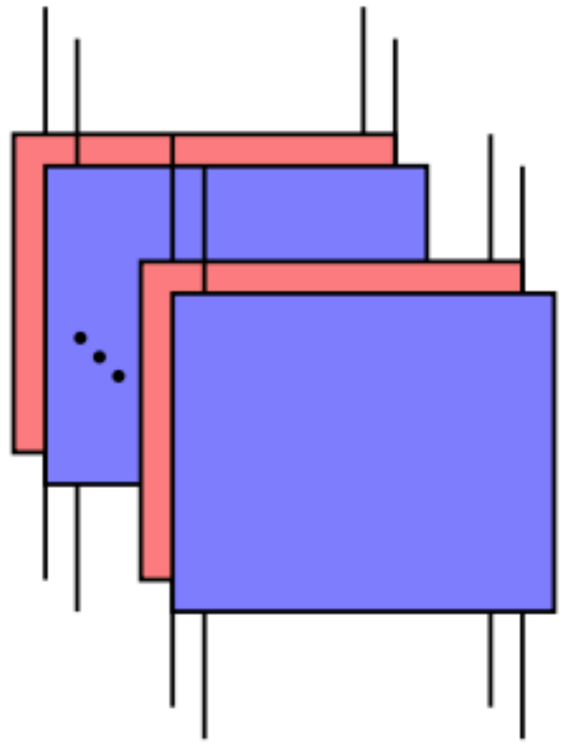
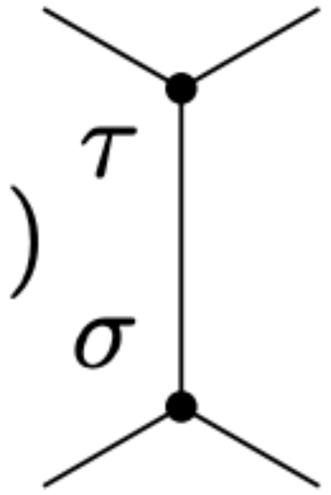




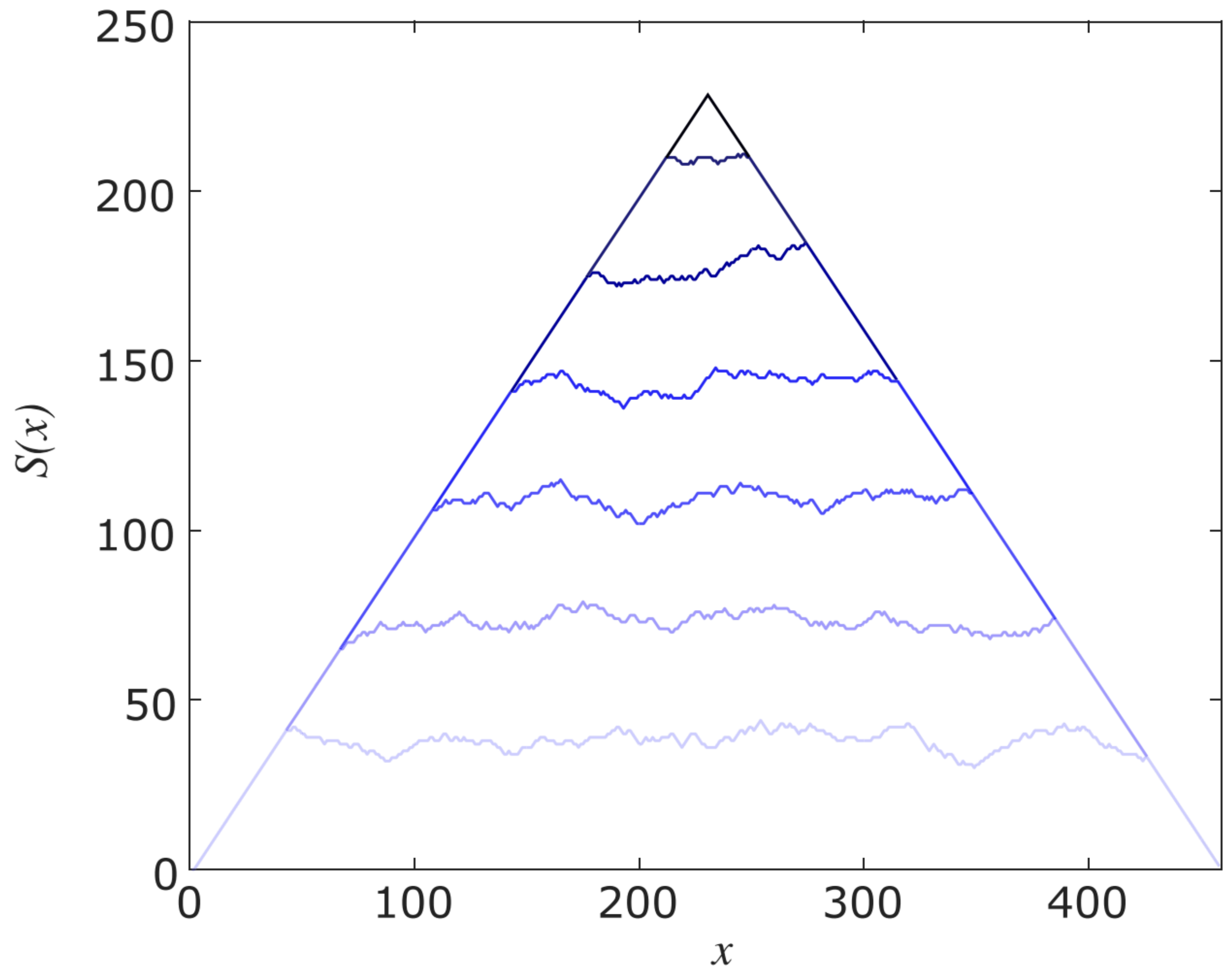
Von Keyserlingk, TR, Pollmann, Sondhi: PRX (2018)

Going to higher moments

$S_n = \frac{1}{1-n} \log \mathrm{Tr}(\rho_A^n)$ ⟶ Want to calculate $Z_n = \overline{\mathrm{Tr}(\rho_A^n)}$ ⟶ Needs $2n$ copies of $U(t)$

Weingarten calculus:  $= \sum_{\tau,\sigma} \mathrm{Wg}(\tau\sigma^{-1})$ 

τ, σ are permutations of n elements



⇒ stat-mech like model with $n!$ states and negative weights

Useful simplification: replace qubits with d-dim. qudits and take $d \gg 1$

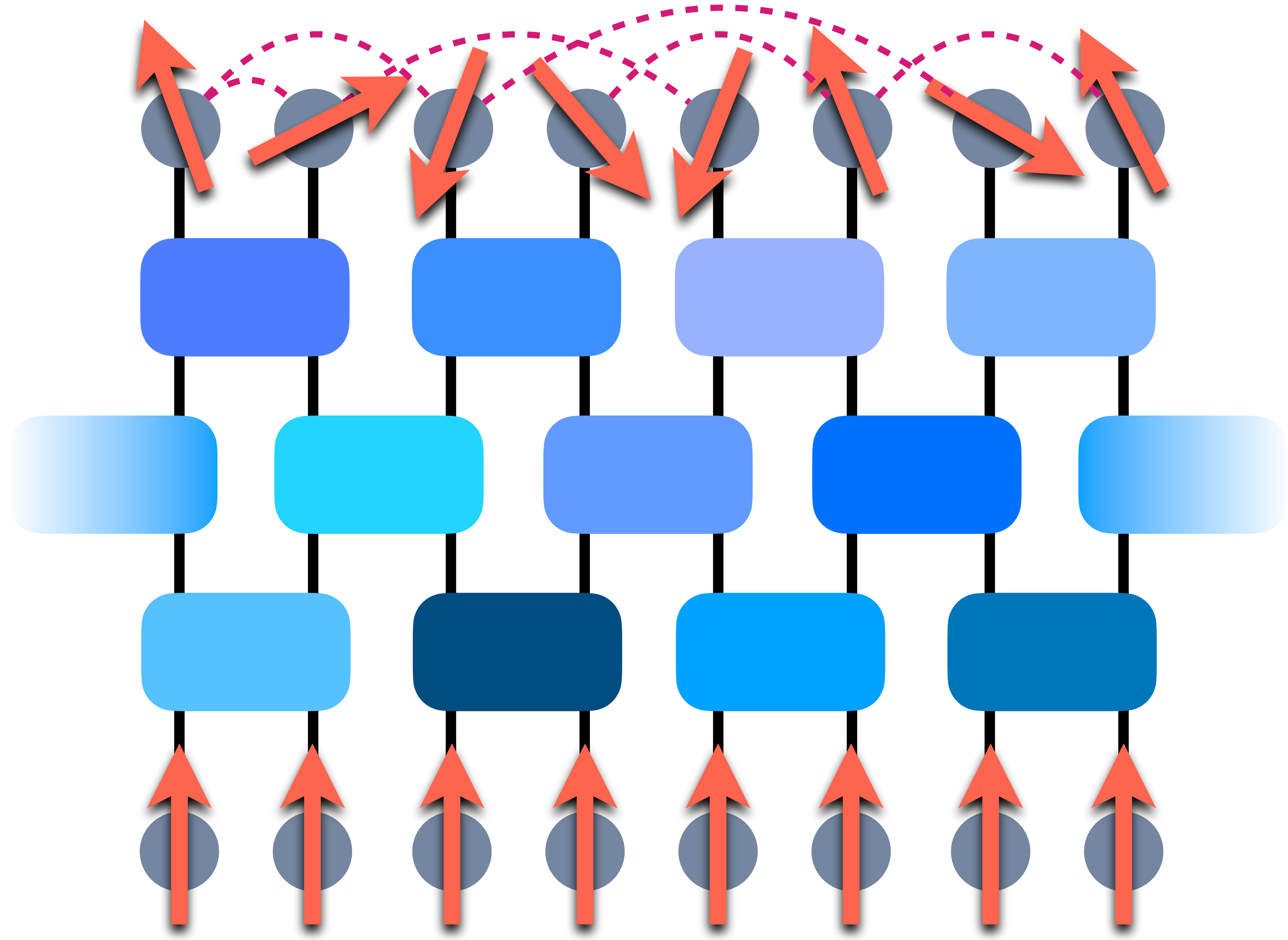
Replica trick: $\overline{S_n} = \frac{1}{1-n} \frac{\partial \overline{Z_n^k}}{\partial k}\Big|_{k=0}$ ⟶ New physics from fluctuations that was absent from $\overline{Z_n}$
Described by Kardar-Parisi-Zhang (KPZ) equation

Zhou, Nahum: PRB (2019) *Nahum, Ruhman, Vijay, Haah: PRX (2017)*

Plan

Introduction: Closed many-body systems far from equilibrium

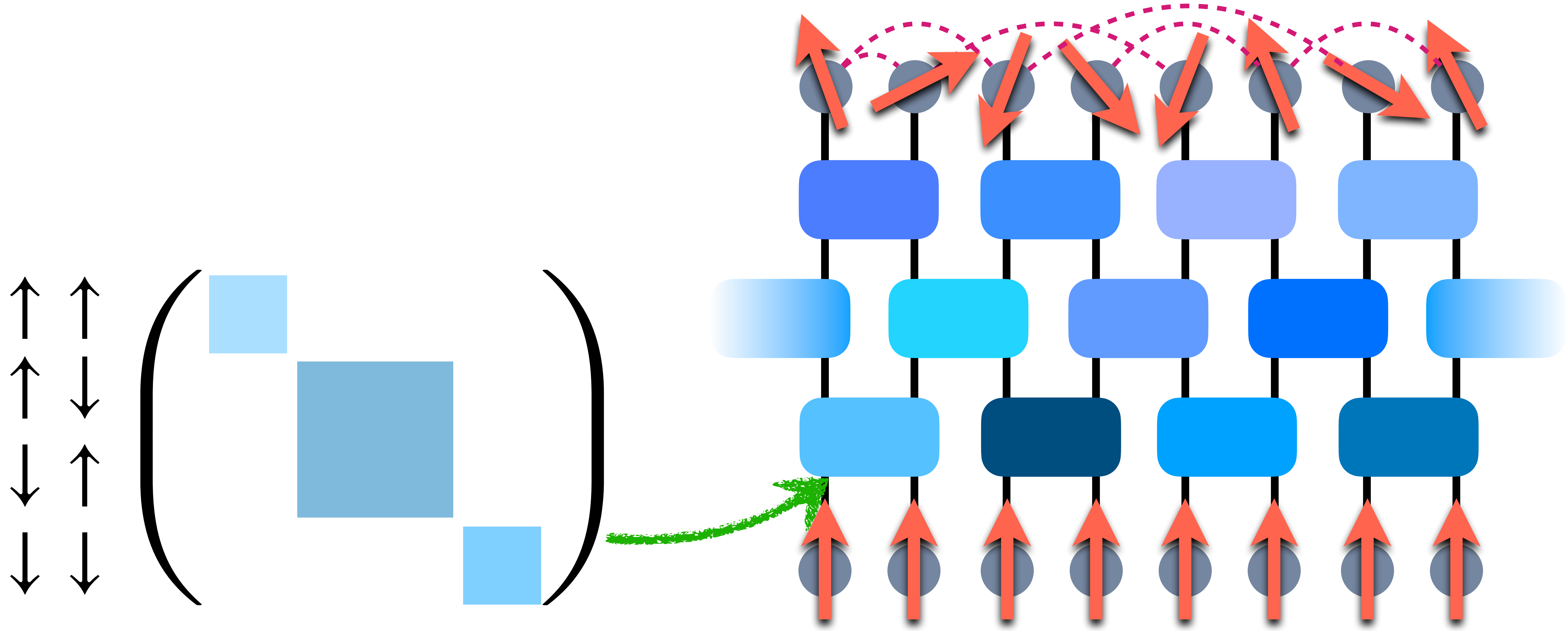
Part 1: Haar random circuits as solvable models



Part 2: Symmetries, measurements and all that

Adding conserved quantities

Want dynamics to conserve $Q = \sum_r \sigma^z_r$

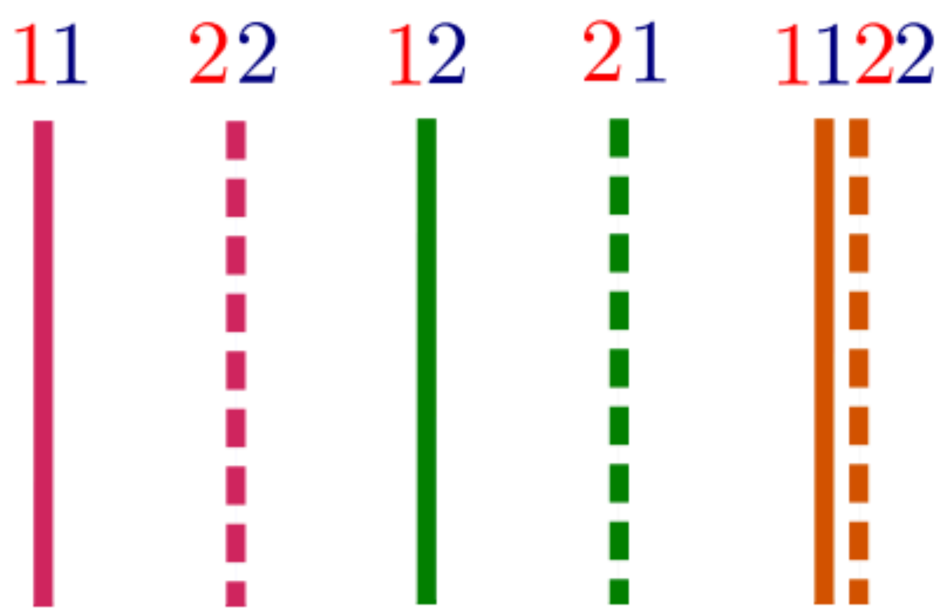


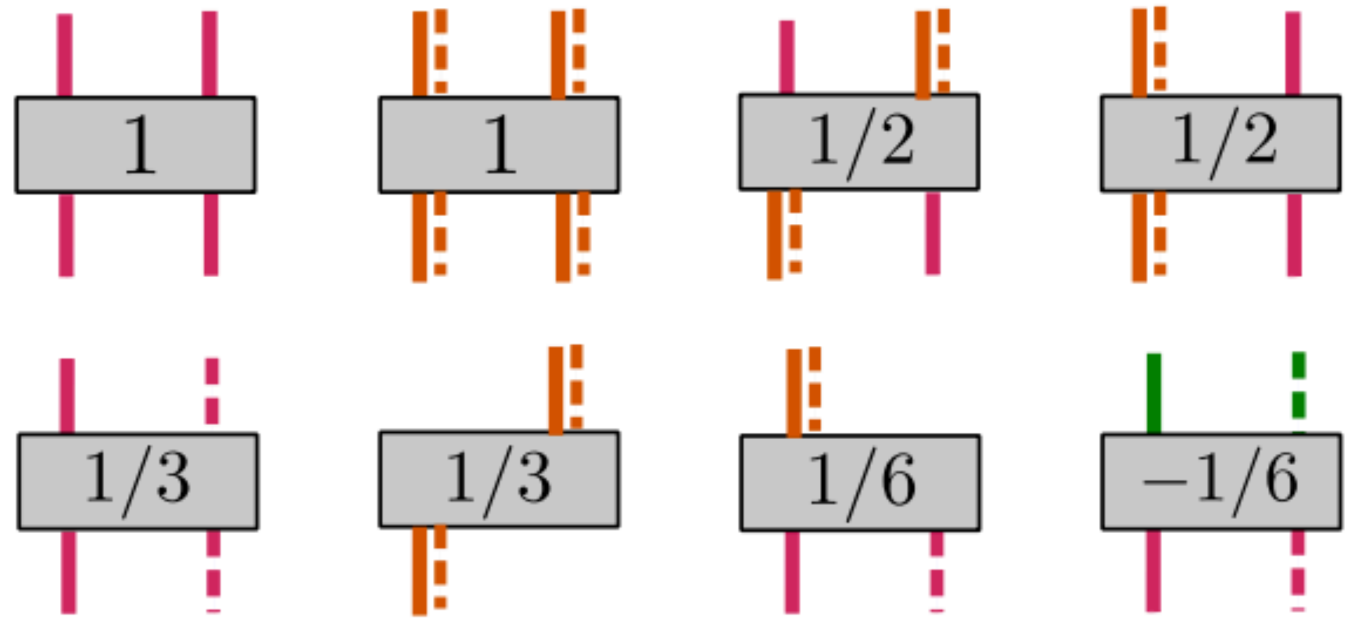
$$\overline{u^\dagger_{r,r+1}\sigma^z_r u_{r,r+1}} = \frac{\sigma^z_r + \sigma^z_{r+1}}{2}$$ random walk

$\Rightarrow$ Diffusion of conserved magnetization

$\overline{u \otimes u^* \otimes u \otimes u^*}$ now leads to partition function with 6 states and negative weights

$\rightarrow$ Turn





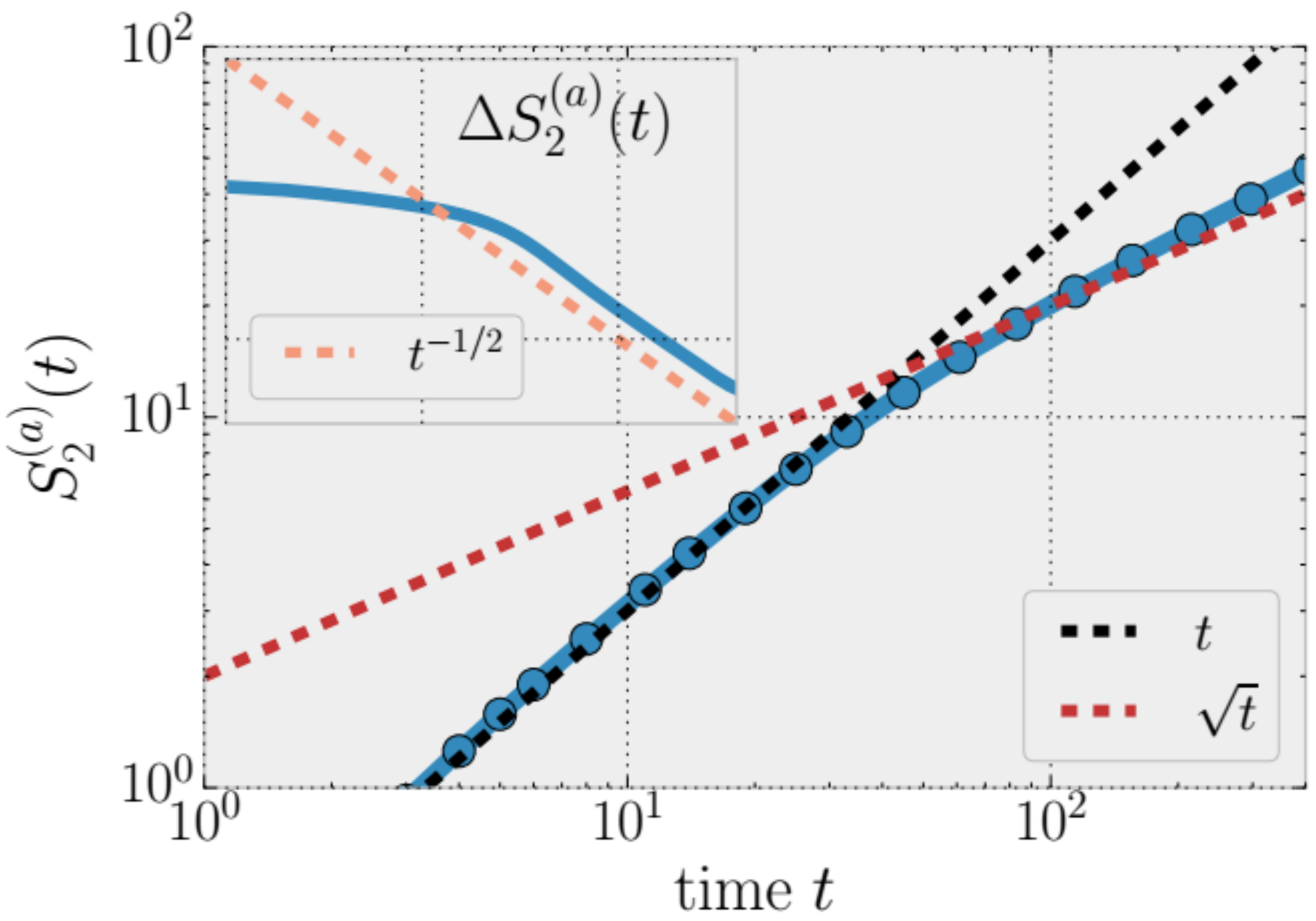
y MPS!

Adding conserved quantities

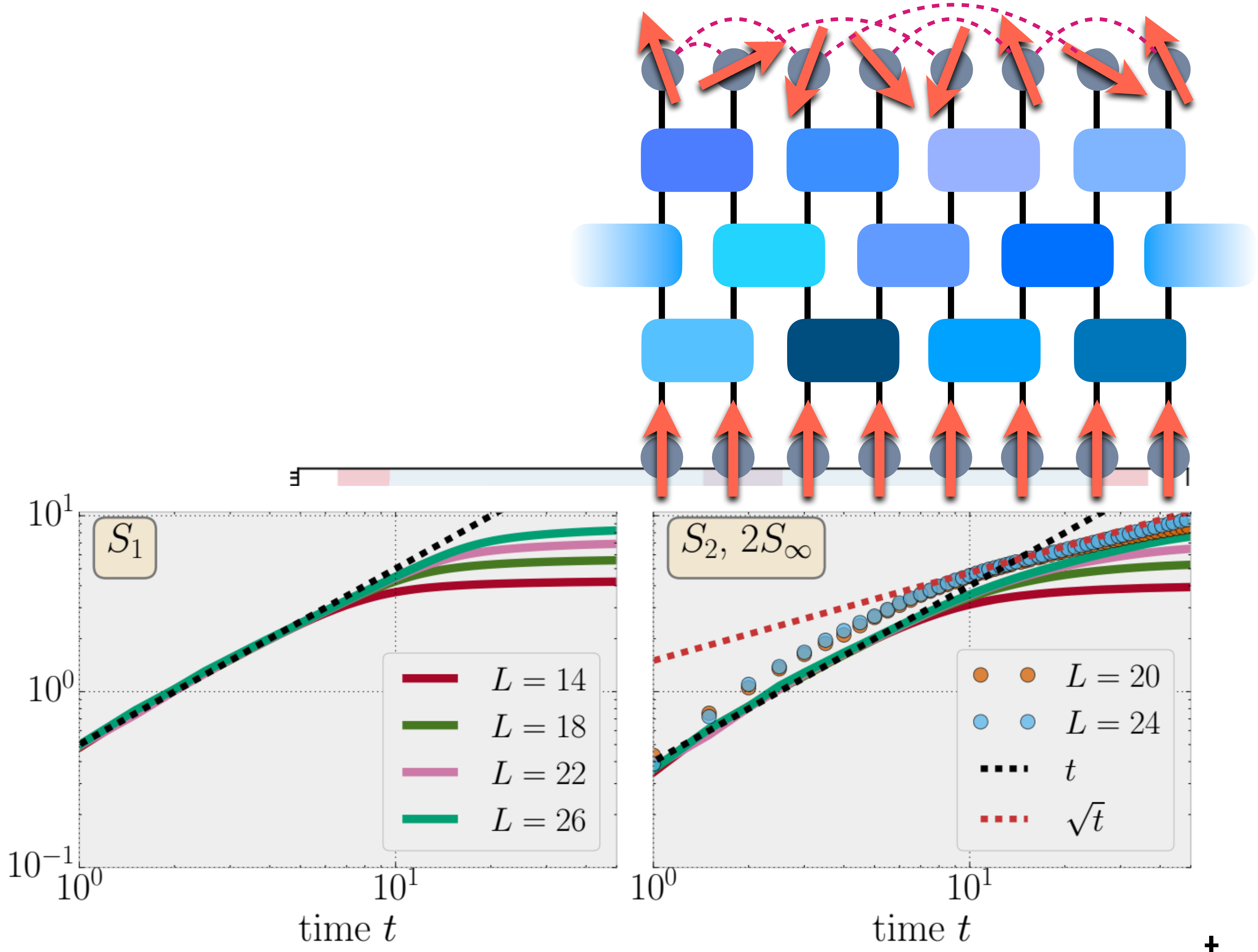
Want dynamics to conserve $Q = \sum_r \sigma_r^z$

Rényi entropies become **diffusive**

$\hat{O}_r(t) = \hat{O}^{\text{diffusive}}(t) + \hat{O}^{\text{ballistic}}(t)$



TR, von Keyserlingk, Pollmann: PRL (2019)



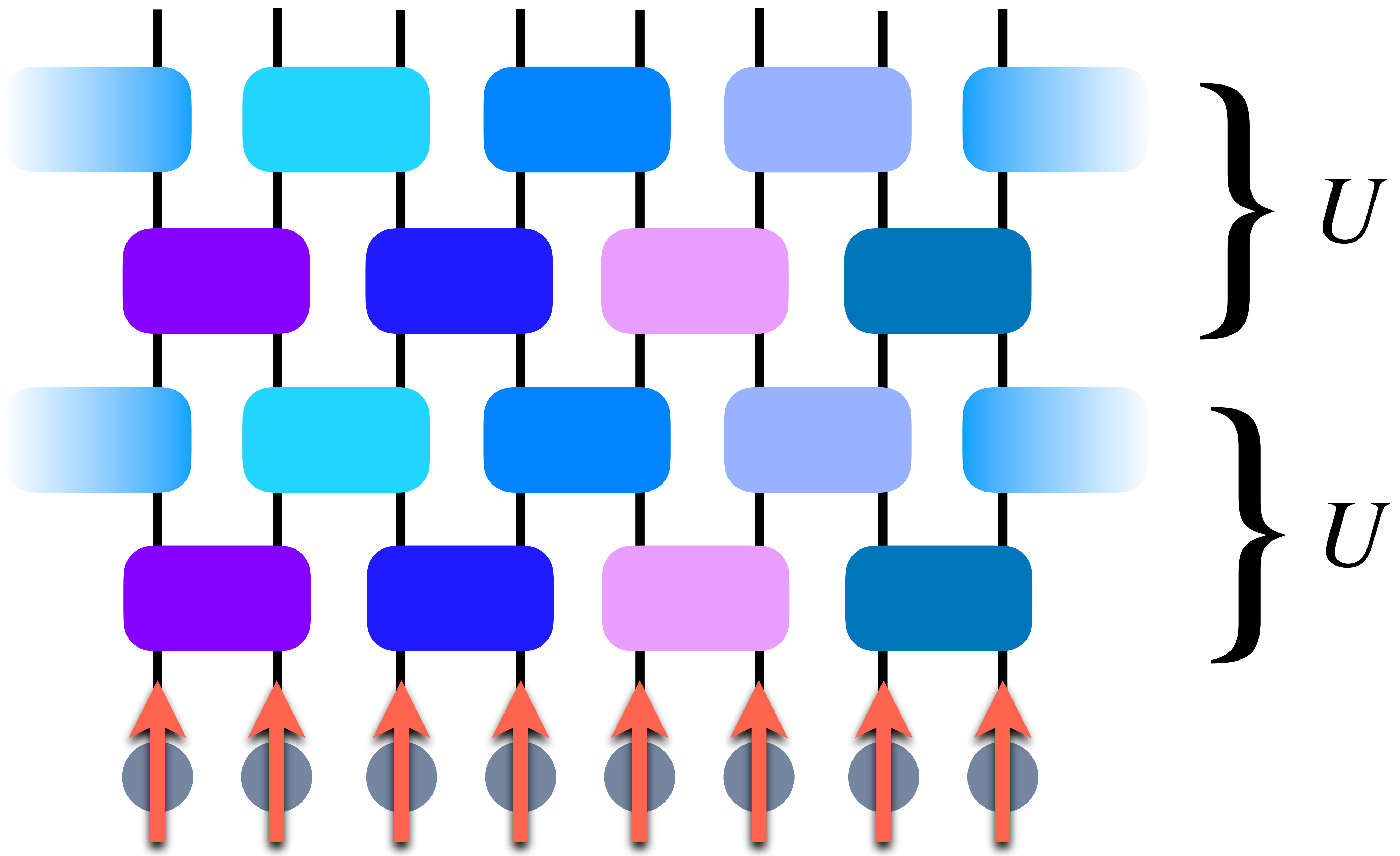
OTOCs develop power law tails behind front

Also seen for Ising spin chain

TR, von Keyserlingk, Pollmann: PRX (2018)

Khemani, Vishwanath, Huse: PRX (2018)

We can also add back (discrete) time translation symmetry

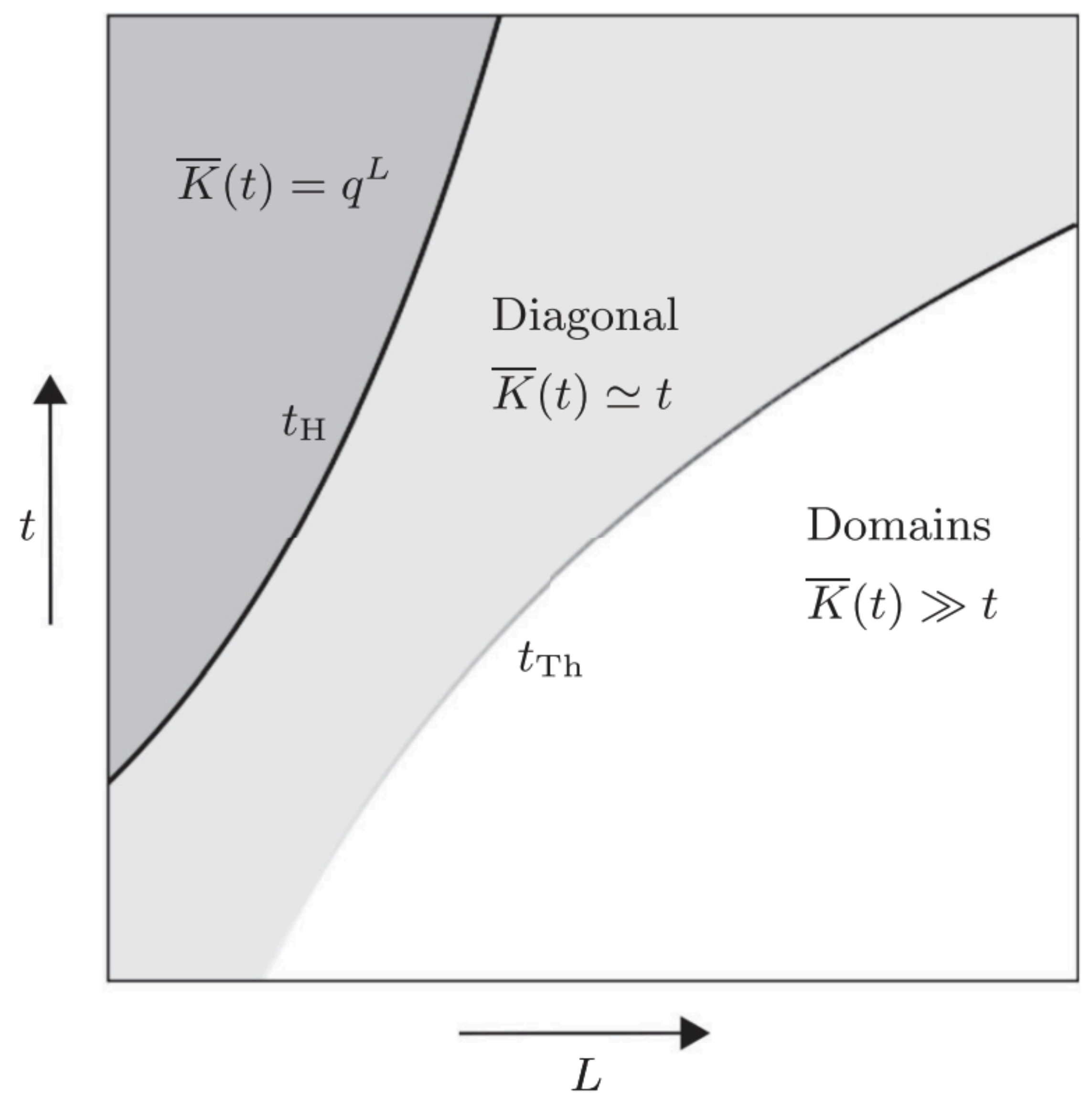


Choose 2 layers randomly then repeat them

Much harder (correlations in time)

But calculations possible as $d \to \infty$

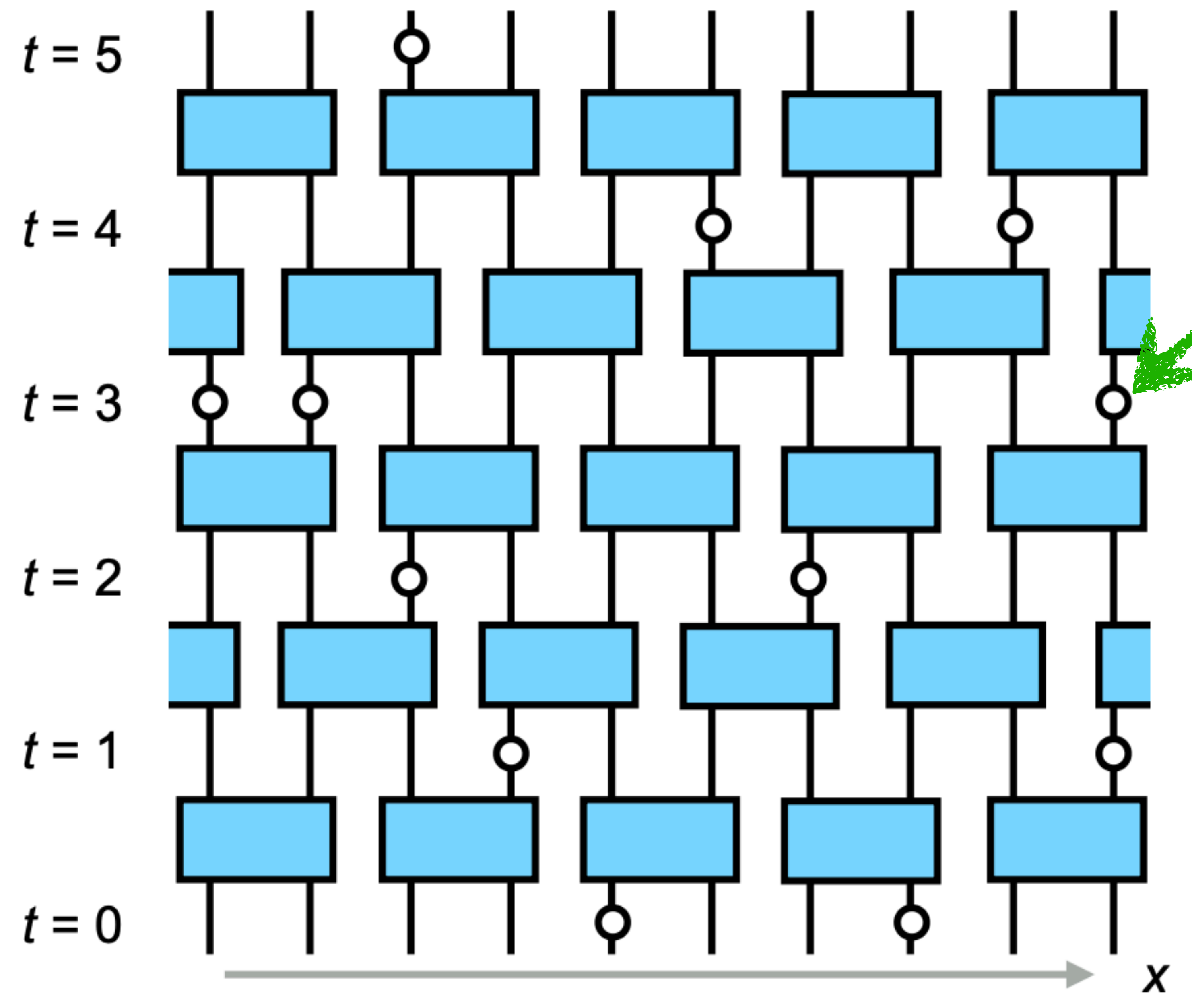
Chan, De Luca, Chalker: PRX (2019)   *Garratt, Chalker: PRX (2021)*



Can calculate **spectral form factor**: $K(t) = |\mathrm{Tr}(U^t)|^2$

→ Reveals the emergence of random matrix spectral statistics

Breaking unitarity with measurements

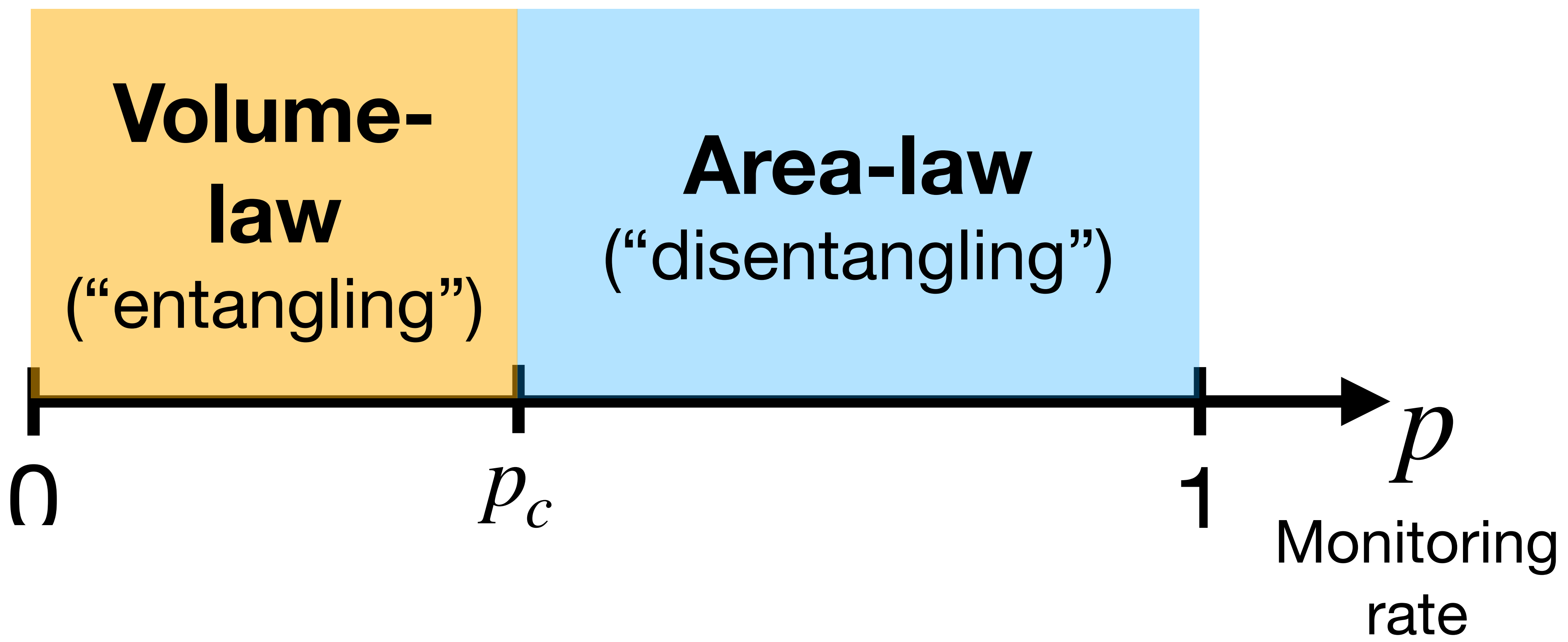


Local spin measurement

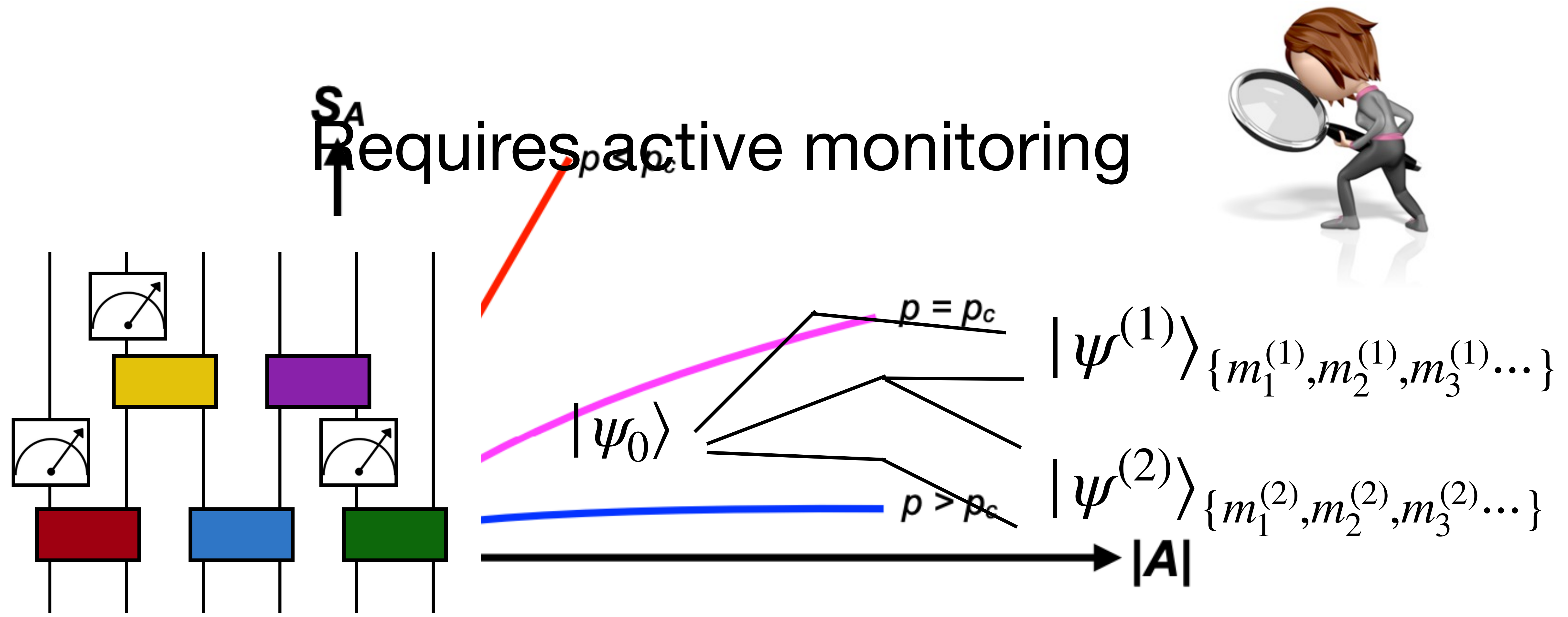
Assume each spin gets measured with probability p at every time step

Leads to **measurement-induced phase transition** in entanglement

Li, Chen, Fisher: PRB (2018) *Nahum, Ruhman, Siknner: PRX (2019)*



Requires active monitoring

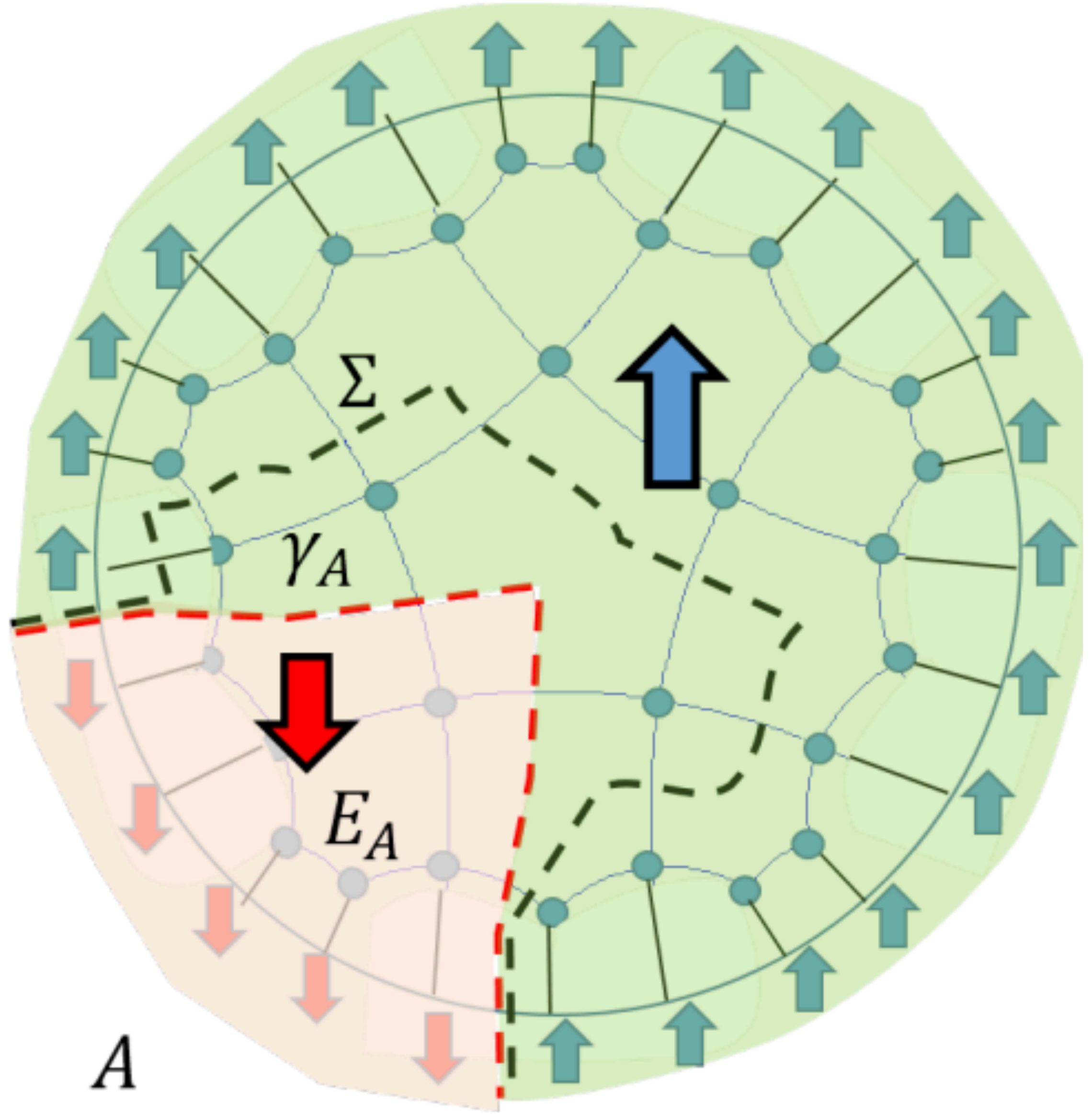


Similar phase transitions occur in random tensor networks

“Holographic” tensor network
(Physical legs on boundary)

Random tensors (column of Haar random u)

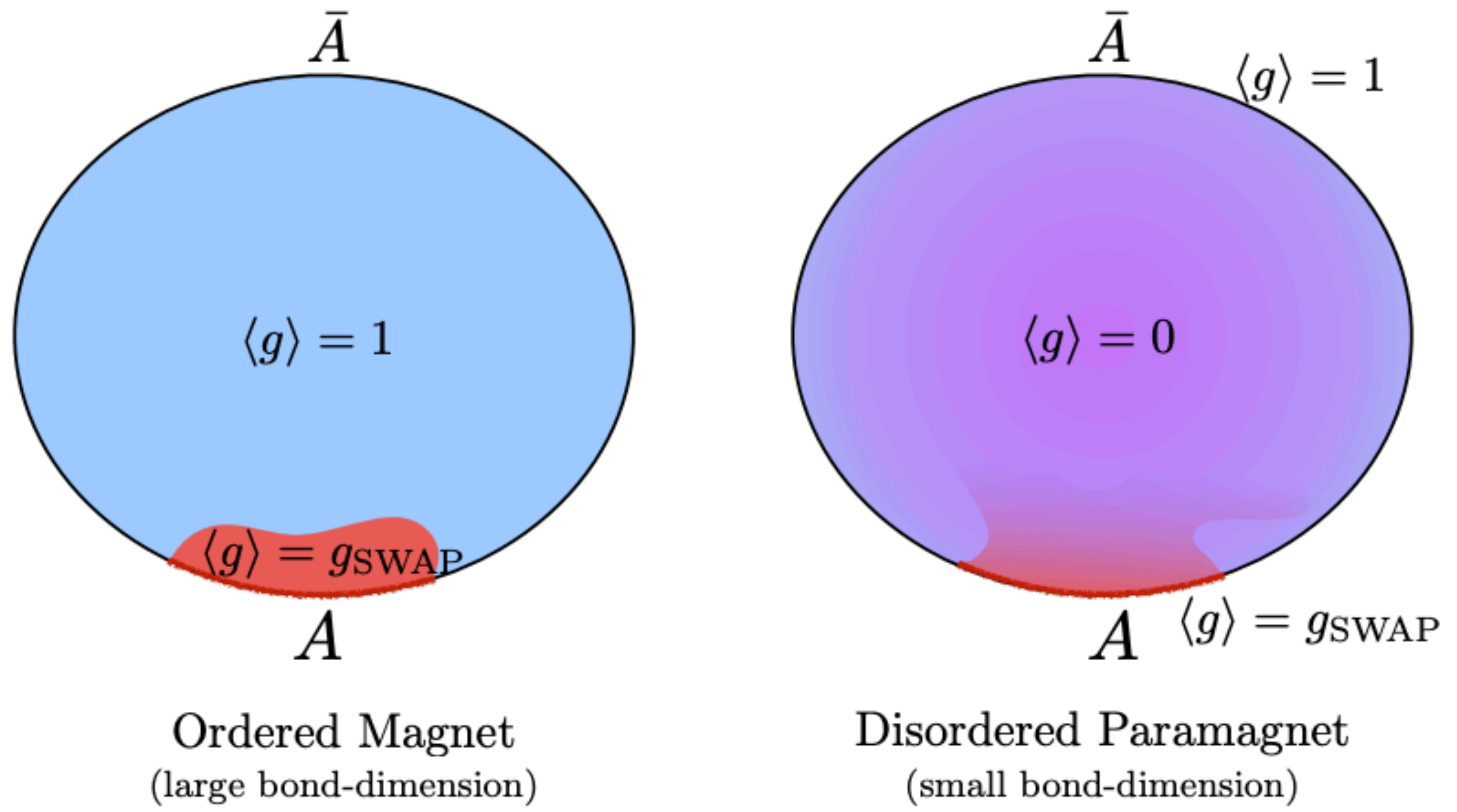
Entanglement: free energy of bulk domain wall



Hayden et al: JHEP (2016)

Large bond dim. → ferromagnet → Ryu-Takayanagi formula

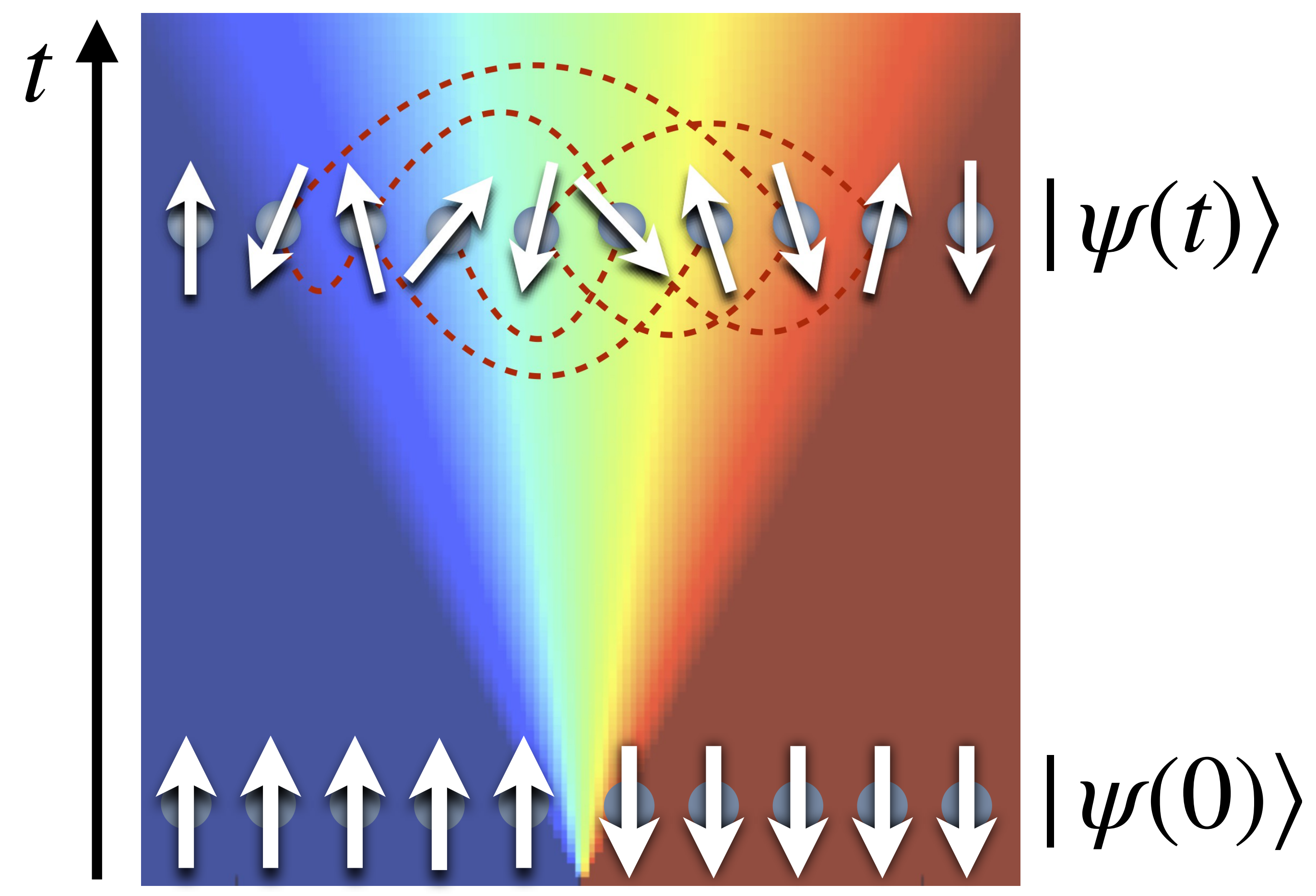
Small bond dim. → paramagnet → RT breaks down



Vasseur et al: PRB (2019)

Summary

- Random circuits provide useful toy models to study quantum dynamics



- Mappings to stat. mech.-like problems (=tensor network contractions)

- We can introduce structure (symmetries, measurements, …) in a controlled way