

Introduction to Matrix Product States

Natalia Chepiga

Kavli Institute of Nanoscience, TU Delft, The Netherlands
⇒ Rudolf Peierls Centre for Theoretical Physics, The University of Oxford, UK

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Scope

- Area law
- Singular value decomposition
- MPO construction
- Variational optimization (finite- and infinite-size DMRG)
- Entanglement entropy and entanglement spectra
- Boundary conditions
- Symmetries and constraints
- Excitations
- Time evolution
- scan-DMRG
- Outlook

Area law

Why tensor networks work?

Area law

Exponential growth of the Hilbert space $\dim H = d^N$
Exact diagonalization is limited to small clusters.

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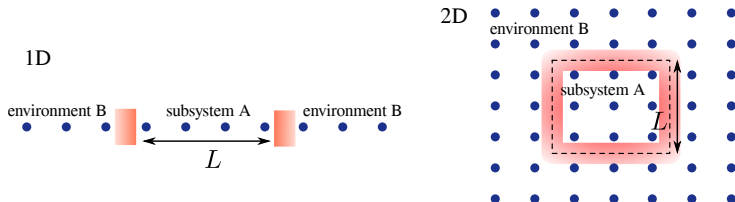
Area law for the entanglement entropy

Low energy states (local H)



Ground states of local
Hamiltonians are less
entangled than a random state
in the Hilbert space

Area law



Entanglement entropy: $S_A = -\text{tr}(\rho_A \log \rho_A)$

GS of local Hamiltonians

Area law: $S_A(L) \propto L^{d-1}$

1D: $S_A(L) = \text{const}$

2D: $S_A(L) \propto L$

Random state

Volume law: $S_A(L) \propto L^d$

Critical state in 1D

$S_A(L) \propto \log(L)$

Area law

Low energy states (local H)



Our goal:

to diagonalize the Hamiltonian
directly in the truncated basis

Number of relevant states $D \propto \exp(S)$

GS of local Hamiltonians

Area law: $S_A(L) \propto L^{d-1}$

1D: $S_A(L) = \text{const}$

2D: $S_A(L) \propto L$

Random state

Volume law: $S_A(L) \propto L^d$

Critical state in 1D

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Graphical notations

 C Number

 A_i Vector

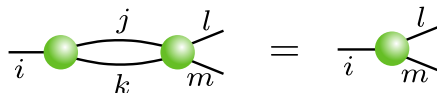
 $A_{i,j}$ Matrix

 $A_{i,j}^{\sigma}$
rank-3 tensor (MPS)

 $A_{i,j}^{\sigma,\sigma'}$
rank-4 tensor (MPO)

 $A_{i,j,k,l}^{\sigma}$
rank-5 tensor (PEPS)

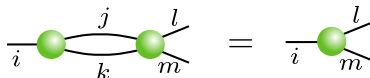
Contraction



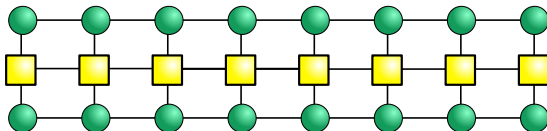
$$\sum_{j,k} A_{i,j,k} B_{j,k,l,m} = T_{i,l,m}$$

- Summation over connected bonds
- Rank of the resulting tensor = number of open legs

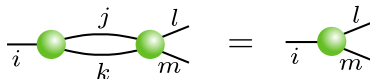
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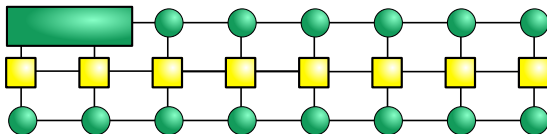
- Complexity $\prod_{i \in \text{connected legs}} D_i \cdot \prod_{j \in \text{open legs}} D_j$
- The order of contraction matters!



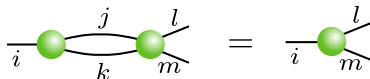
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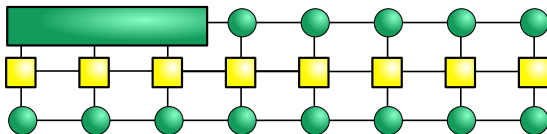
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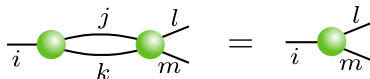
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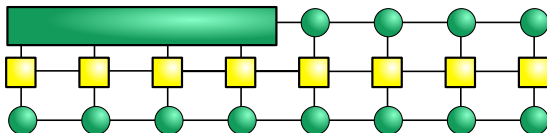
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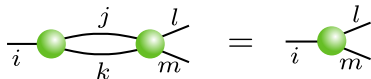
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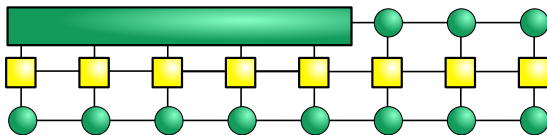
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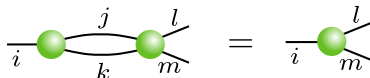


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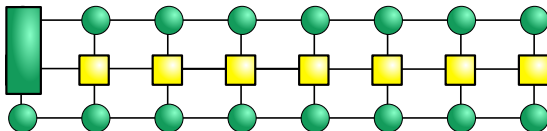


Exponential growth of complexity!

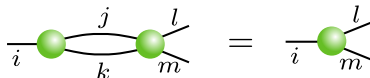
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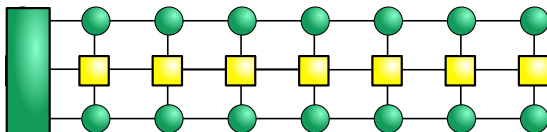
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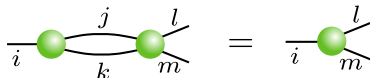
Contraction



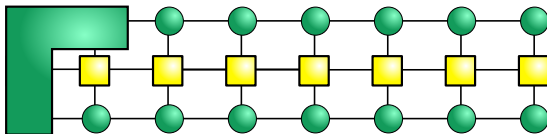
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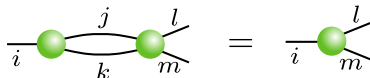
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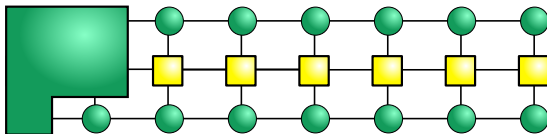
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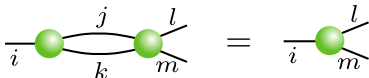
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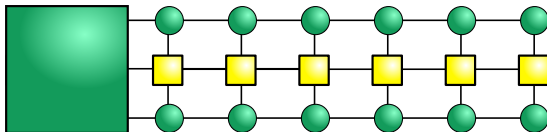
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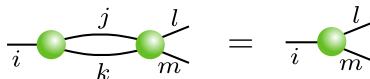
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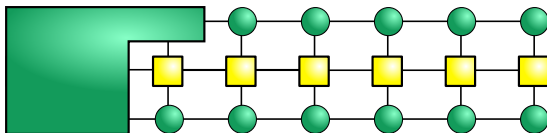
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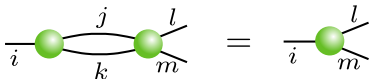
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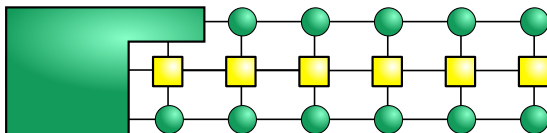
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Contraction



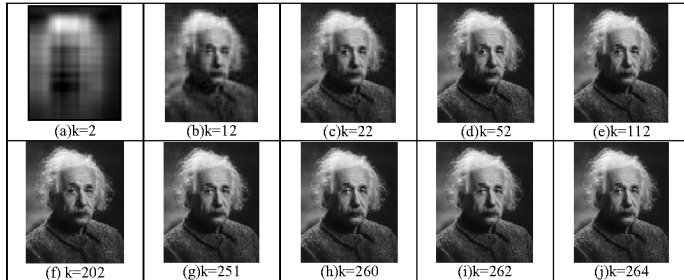
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Complexity stays finite!

SVD

singular values decomposition



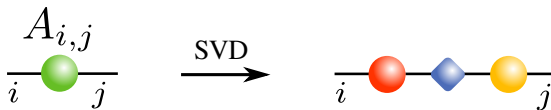
Singular Values Decomposition (SVD)

For any rectangular matrix $M_{i,j}$ exists a decomposition

$$M = U_{i,k} S_{k,k} V_{k,j}^\dagger$$

such that:

- $U^\dagger U = \mathbb{I}$
- S is a diagonal matrix with non-negative entries
- $V^\dagger V = \mathbb{I}$



Schmidt decomposition

- Quantum state:

$$|\psi\rangle = \sum_{i,j} \Psi_{i,j} |i\rangle_A |j\rangle_B,$$

where $|i\rangle_A$ and $|j\rangle_B$ are orthonormal basis of subsystems A and B.

- Treat $\Psi_{i,j}$ as a matrix and perform SVD
- Schmidt decomposition

$$|\psi\rangle = \sum_{i,j} \sum_k U_{i,k} S_{k,k} V_{k,j}^\dagger |i\rangle_A |j\rangle_B$$

Schmidt decomposition

- Quantum state:

$$|\psi\rangle = \sum_{i,j} \Psi_{i,j} |i\rangle_A |j\rangle_B,$$

where $|i\rangle_A$ and $|j\rangle_B$ are orthonormal basis of subsystems A and B.

- Treat $\Psi_{i,j}$ as a matrix and perform SVD
- Area law - D relevant states only

$$|\psi\rangle = \sum_{i,j} \sum_{\substack{D \\ k}} U_{i,k} S_{k,k} V_{k,j}^\dagger |i\rangle_A |j\rangle_B$$

Some states have exact MPS representations

Majumdar-Ghosh chain:

$$H = \sum_i J_1 \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{J_2}{2} \mathbf{S}_i \cdot \mathbf{S}_{i+2} \quad (1)$$

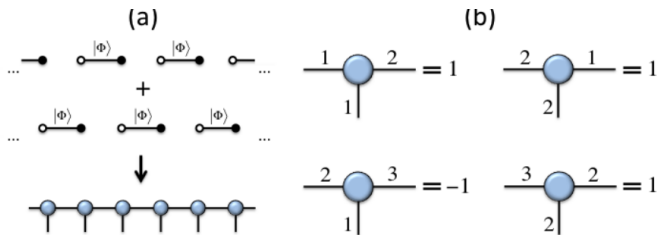
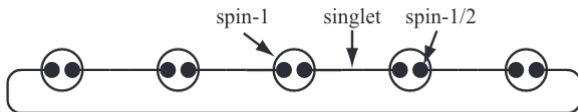


Figure 24: (color online) MPS for the Majumdar-Ghosh state: (a) the superposition of two dimerized states of singlets $|\Phi\rangle$ in (a) can be written in terms of an infinite MPS with 1-site unit cell, with non-zero coefficients as in (b).

Some states have exact MPS representations

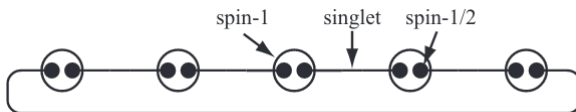
AKLT :

$$H = \sum_i J_1 \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{J_b}{3} (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 \quad (2)$$



Some states have exact MPS representations

$$\text{AKLT :} \quad H = \sum_i J_1 \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{J_b}{3} (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 \quad (3)$$



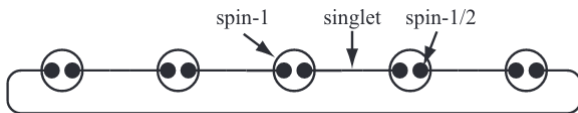
$$\Sigma = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

$$M^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M^0 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad M^- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$|\psi\rangle = \sum_{\sigma} \text{Tr}(M^{\sigma_1} \Sigma M^{\sigma_2} \Sigma \dots M^{\sigma_L} \Sigma) |\sigma\rangle,$$

Some states have exact MPS representations

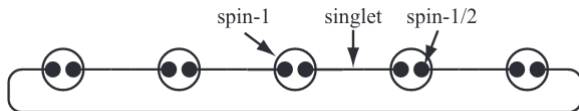
$$\text{AKLT :} \quad H = \sum_i J_1 \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{J_b}{3} (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 \quad (4)$$



$$\tilde{A}^+ = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad \tilde{A}^0 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & +\frac{1}{2} \end{bmatrix} \quad \tilde{A}^- = \begin{bmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Some states have exact MPS representations

$$\text{AKLT :} \quad H = \sum_i J_1 \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{J_b}{3} (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 \quad (5)$$



$$\tilde{A}^+ = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad \tilde{A}^0 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & +\frac{1}{2} \end{bmatrix} \quad \tilde{A}^- = \begin{bmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Interestingly, often (always?) exact MPS states corresponds to a disorder point signaling the appearance of incommensurability

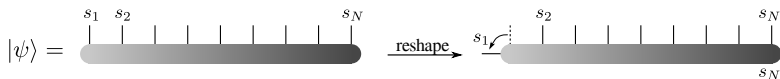
Bring quantum states into MPS

$$\begin{aligned}
 c_{\sigma_1 \dots \sigma_L} &= \Psi_{(\sigma_1 \dots \sigma_{L-1}), \sigma_L} \\
 &= \sum_{a_{L-1}} U_{(\sigma_1 \dots \sigma_{L-1}), a_{L-1}} S_{a_{L-1}, a_{L-1}} (V^\dagger)_{a_{L-1}, \sigma_L} \\
 &= \sum_{a_{L-1}} \Psi_{(\sigma_1 \dots \sigma_{L-2}), (\sigma_{L-1} a_{L-1})} B_{a_{L-1}}^{\sigma_L} \\
 &= \sum_{a_{L-2}, a_{L-1}} U_{(\sigma_1 \dots \sigma_{L-2}), a_{L-2}} S_{a_{L-2}, a_{L-2}} (V^\dagger)_{a_{L-2}, (\sigma_{L-1} a_{L-1})} B_{a_{L-1}}^{\sigma_L} \\
 &= \sum_{a_{L-2}, a_{L-1}} \Psi_{(\sigma_1 \dots \sigma_{L-3}), (\sigma_{L-2} a_{L-2})} B_{a_{L-2}, a_{L-1}}^{\sigma_{L-1}} B_{a_{L-1}}^{\sigma_L} = \dots \\
 &= \sum_{a_1, \dots, a_{L-1}} B_{a_1}^{\sigma_1} B_{a_1, a_2}^{\sigma_2} \dots B_{a_{L-2}, a_{L-1}}^{\sigma_{L-1}} B_{a_{L-1}}^{\sigma_L}.
 \end{aligned}$$

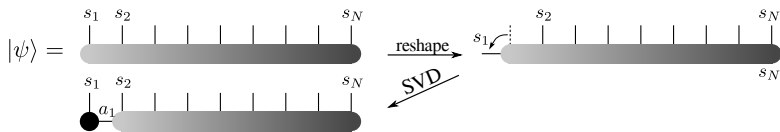
Bring quantum states into MPS

$$|\psi\rangle = \text{---} \overset{s_1}{\text{---}} \overset{s_2}{\text{---}} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \overset{s_N}{\text{---}}$$

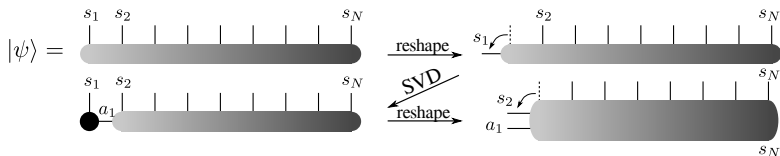
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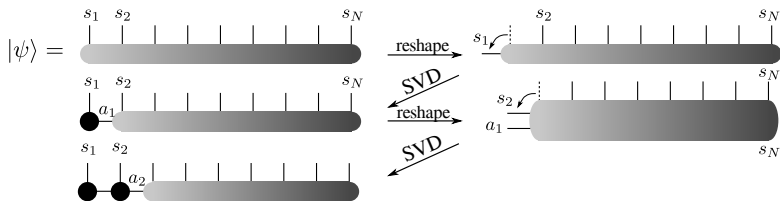
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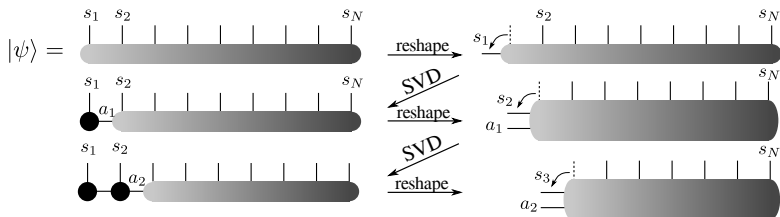
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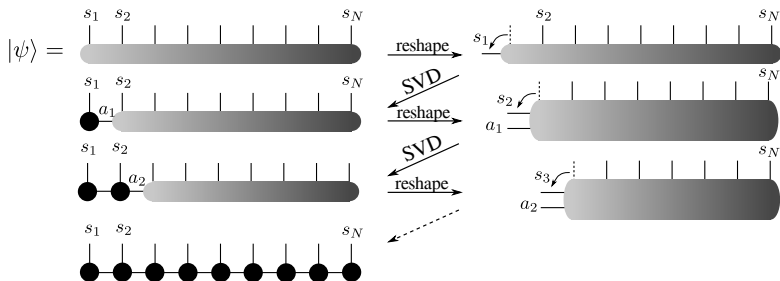
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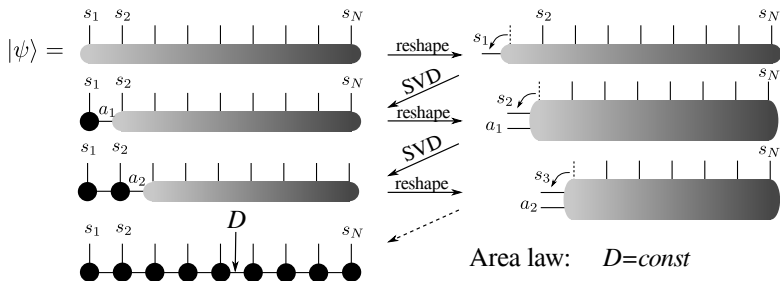
Bring quantum states into MPS



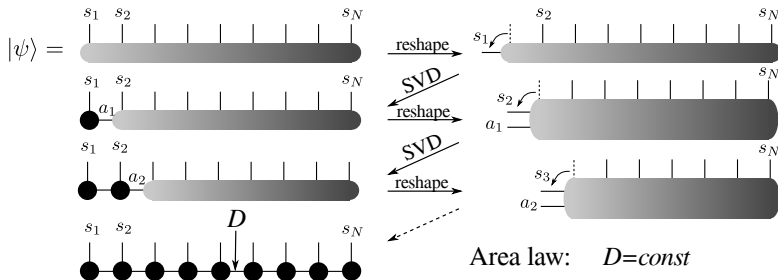
Bring quantum states into MPS



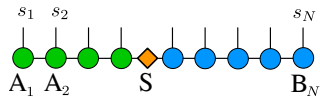
Bring quantum states into MPS



Bring quantum states into MPS



Mixed-canonical form



Normalization:



Normalization

The goal is to find $|\Psi\rangle$ that minimizes the energy:

$$E = \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

If norm is fixed $\langle \Psi | \Psi \rangle = 1$, it becomes

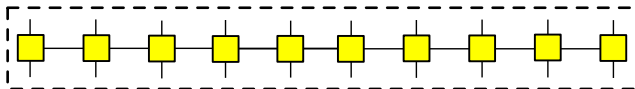
$$E = \langle \Psi | \hat{H} | \Psi \rangle$$

In variational optimization a **generalized** eigenvalue problem is reduced to a **generalized** eigenvalue problem:

$$\hat{H}_{eff}|\psi\rangle = E|\psi\rangle \quad \text{instead of} \quad \hat{H}_{eff}|\psi\rangle = E\hat{N}_{eff}|\psi\rangle$$

MPO

Matrix Product Operators



$d^N \times d^N$
Hamiltonian

MPO construction

$$H = \sum_j J(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + BS_j^z$$

For a given site j write all possible terms in the Hamiltonian:

$$\begin{array}{ccc}
 I \dots I & BS_j^z & I \dots I \\
 I \dots I JS_{j-1}^x & S_j^x & I \dots I \\
 I \dots I JS_{j-1}^y & S_j^y & I \dots I \\
 I \dots I & JS_j^x & S_{j+1}^x I \dots I \\
 I \dots I & JS_j^y & S_{j+1}^y I \dots I \\
 I \dots JS_i^x S_{i+1}^x \dots I & I & I \dots I \\
 I \dots JS_i^y S_{i+1}^y \dots I & I & I \dots I \\
 I \dots BS_i^z \dots I & I & I \dots I \\
 I \dots I & I & I \dots JS_i^x S_{i+1}^x \dots I \\
 I \dots I & I & I \dots JS_i^y S_{i+1}^y \dots I \\
 I \dots I & I & I \dots BS_i^z \dots I
 \end{array}$$

MPO construction

$$H = \sum_j J(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + BS_j^z$$

For a given site j write all possible terms in the Hamiltonian:

$I \dots I$	BS_j^z	$I \dots I$
$I \dots I JS_{j-1}^x$	S_j^x	$I \dots I$
$I \dots I JS_{j-1}^y$	S_j^y	$I \dots I$
$I \dots I$	JS_j^x	$S_{j+1}^x I \dots I$
$I \dots I$	JS_j^y	$S_{j+1}^y I \dots I$
$I \dots JS_i^x S_{i+1}^x \dots I$	I	$I \dots I$
$I \dots JS_i^y S_{i+1}^y \dots I$	I	$I \dots I$
$I \dots BS_i^z \dots I$	I	$I \dots I$
$I \dots I$	I	$I \dots JS_i^x S_{i+1}^x \dots I$
$I \dots I$	I	$I \dots JS_i^y S_{i+1}^y \dots I$
$I \dots I$	I	$I \dots BS_i^z \dots I$

MPO construction

$$H = \sum_j J(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + BS_j^z$$

For a given site j write all possible terms in the Hamiltonian:

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$I \dots I JS_{j-1}^x$	S_j^x	$I \dots I$
$I \dots I JS_{j-1}^y$	S_j^y	$I \dots I$
$I \dots I$	JS_j^x	$S_{j+1}^x I \dots I$
$I \dots I$	JS_j^y	$S_{j+1}^y I \dots I$
$I \dots \text{Full} \dots I$	I	$I \dots I$
$I \dots I$	I	$I \dots \text{Full} \dots I$

- Seven non-zero entries in the MPO

MPO construction

$$H = \sum_j J(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + BS_j^z$$

For a given site j write all possible terms in the Hamiltonian:

$$\begin{array}{c|c|c}
 I \dots I & BS_j^z & I \dots I \\
 I \dots I JS_{j-1}^x & S_j^x & I \dots I \\
 I \dots I JS_{j-1}^y & S_j^y & I \dots I \\
 I \dots I & JS_j^x & S_{j+1}^x I \dots I \\
 I \dots I & JS_j^y & S_{j+1}^y I \dots I \\
 I \dots \text{Full} \dots I & I & I \dots I \\
 I \dots I & I & I \dots \text{Full} \dots I
 \end{array}$$

- Seven non-zero entries in the MPO
- Look at the left and right basis in which the MPO is going to be written

MPO construction

- For a given site j write all possible terms in the Hamiltonian:
- Seven non-trivial entries in the MPO
- Look at the left and right basis in which the MPO is going to be written

$$\begin{array}{c|cccc} I \dots \text{Full} \dots I & & & & \\ I \dots I J S_{j-1}^x & & & & \\ I \dots I J S_{j-1}^y & & & & \\ I \dots I & & & & \\ \hline & I \dots I & S_{j+1}^x I \dots I & S_{j+1}^y I \dots I & I \dots \text{Full} \dots I \end{array}$$

MPO construction

- For a given site j write all possible terms in the Hamiltonian:
- Five non-trivial entries in the MPO
- Look at the left and right basis in which the MPO is going to be written
- Fill-in the matrix:

$I \dots \text{Full} \dots I$	I	0	0	0
$I \dots I J S_{j-1}^x$	S_j^x	0	0	0
$I \dots I J S_{j-1}^y$	S_j^y	0	0	0
$I \dots I$	$B S_j^z$	$J S_j^x$	$J S_j^y$	I
	$I \dots I$	$S_{j+1}^x I \dots I$	$S_{j+1}^y I \dots I$	$I \dots \text{Full} \dots I$

Bulk MPO. What about edges?

$$H = \sum_j J(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + B S_j^z$$

$$H_j = \begin{pmatrix} I & \cdot & \cdot & \cdot \\ S_j^x & \cdot & \cdot & \cdot \\ S_j^y & \cdot & \cdot & \cdot \\ B S_j^z & J S_j^x & J S_j^y & I \end{pmatrix}$$

MPO construction

$$H = \sum_j J(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + BS_j^z$$

Bulk MPO:

$$H_j = \begin{pmatrix} I & \cdot & \cdot & \cdot \\ S_j^x & \cdot & \cdot & \cdot \\ S_j^y & \cdot & \cdot & \cdot \\ BS_j^z & JS_j^x & JS_j^y & I \end{pmatrix}$$

First and last sites:

$$H_1 = \begin{pmatrix} BS_1^z & JS_1^x & JS_1^y & I \end{pmatrix} \quad H_j = \begin{pmatrix} I \\ S_j^x \\ S_j^y \\ BS_j^z \end{pmatrix}$$

Exponentially decaying interactions

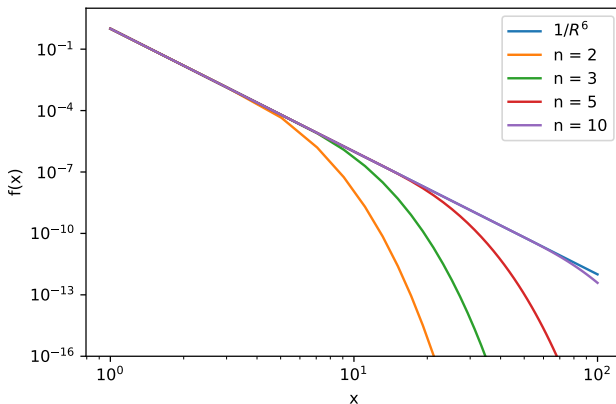
- Long-range interactions are in general difficult
- ... but there is one exception - exponentially decaying potential

$$H = \sum_j J(r) S_j^z S_{j+1}^z \quad \text{with} \quad J(r) = J e^{-r/\xi} = J \lambda^r$$

$$H_j = \begin{pmatrix} I & \cdot & \cdot \\ S^z & \lambda S_j^z & \cdot \\ \cdot & J \lambda S_j^z & I \end{pmatrix}$$

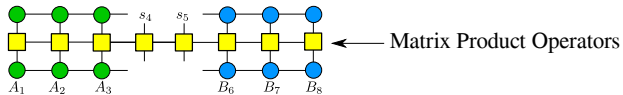
Exponentially decaying interactions

- Long-range interactions are in general difficult
- ... but there is one exception - exponentially decaying potential
- Approximate long-range interactions with exponentials

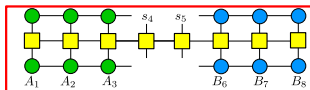


DMRG / variational MPS

DMRG sweep



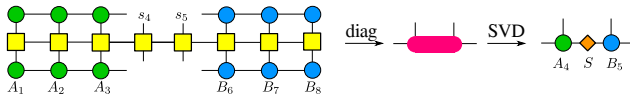
DMRG sweep



Group the legs and treat this
rank-8 tensor as a matrix

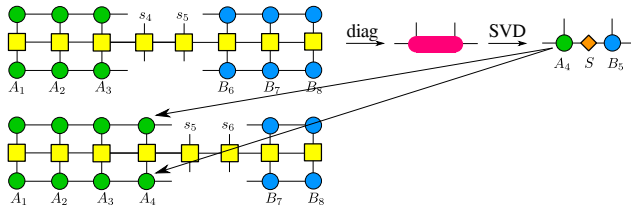
DMRG sweep

Left-to-right:



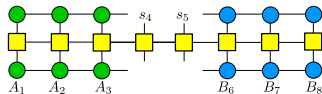
DMRG sweep

Left-to-right:



DMRG sweep

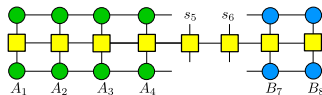
Left-to-right:



diag



SVD



diag

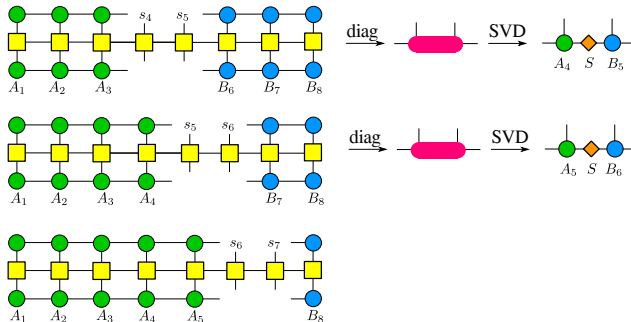


SVD



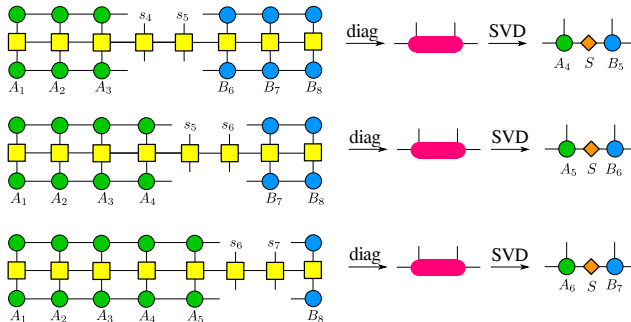
DMRG sweep

Left-to-right:



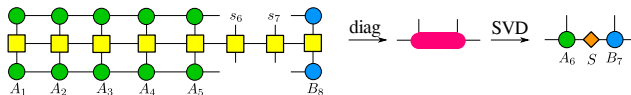
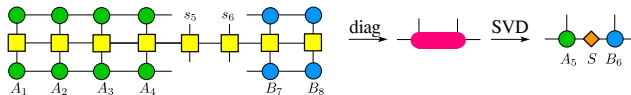
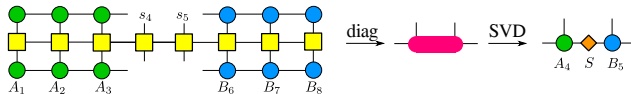
DMRG sweep

Left-to-right:

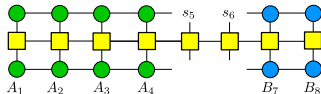


DMRG sweep

Left-to-right:

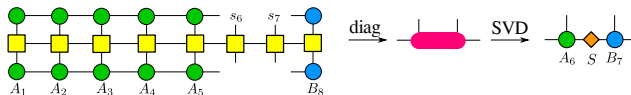
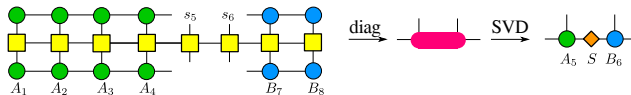
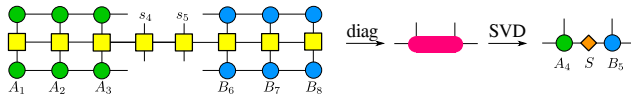


Right-to-left:

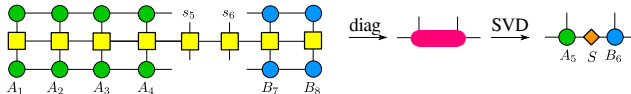


DMRG sweep

Left-to-right:

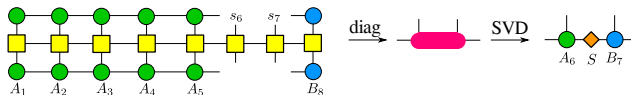
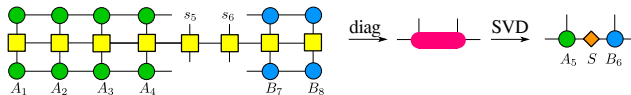
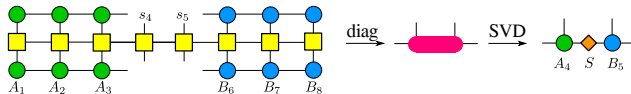


Right-to-left:

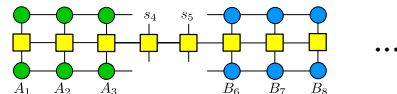
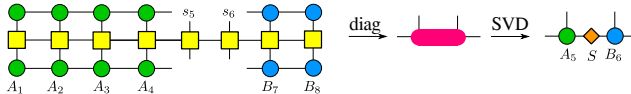


DMRG sweep

Left-to-right:



Right-to-left:



Convergence criteria

Convergence criteria

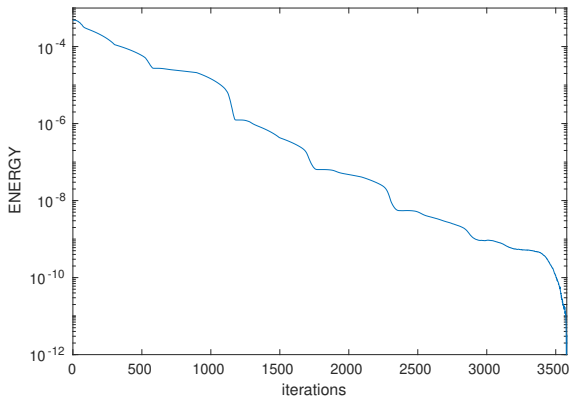
- Energy
- Entanglement
- Observable of interest:
 - Local magnetization/density
 - Dimerization / local correlations
 - Distant correlations $\langle O_i O_j \rangle$
 - Extreme values (disorder)
 - ...

Convergence criteria

- Local magnetization/density
- Dimerization / local oscillations
- Extreme values (disorder)
- Energy
- Distant correlations $\langle O_i O_j \rangle$
- Entanglement

Convergence criteria

Typical energy convergence:



DMRG is variational!

Convergence criteria

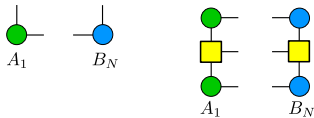
- Local magnetization/density
- Dimerization / local oscillations
- Extreme values (disorder)
- Energy
- Distant correlations $\langle O_i O_j \rangle$
- Entanglement

DMRG has to converge with respect to BOTH
the bond dimension D and the number of iterations!

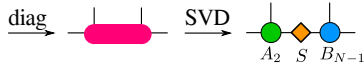
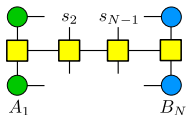
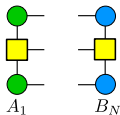
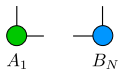
Initial guess

- Product state
- Random state
- Infinite-size DMRG

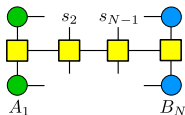
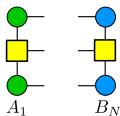
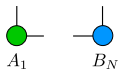
Infinite-size DMRG



Infinite-size DMRG



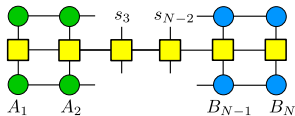
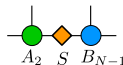
Infinite-size DMRG



diag



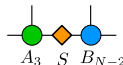
SVD



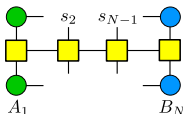
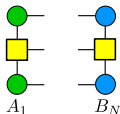
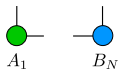
diag



SVD



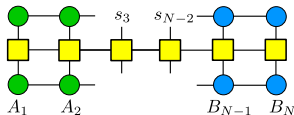
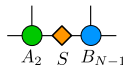
Infinite-size DMRG



diag



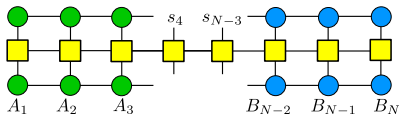
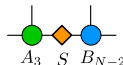
SVD



diag



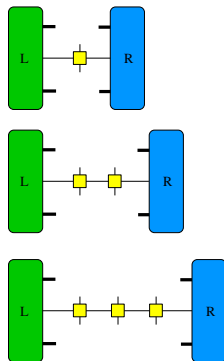
SVD



...

One-, two- and multi-site DMRG

?-site DMRG



- Traditionally DMRG is a two-site algorithm.
- Single-site is cheaper by a factor d^2
- Naively the bond dimension in one-site DMRG is fixed
- Now there are efficient ways to increase it:
 - Hubig et al (PRB 2015)
 - von Delft et al (PRL 2022)
- Multi-site provide stability but they are expensive, though there are tricks like DMRG²

Lanczos algorithm

Iterative eigensolver

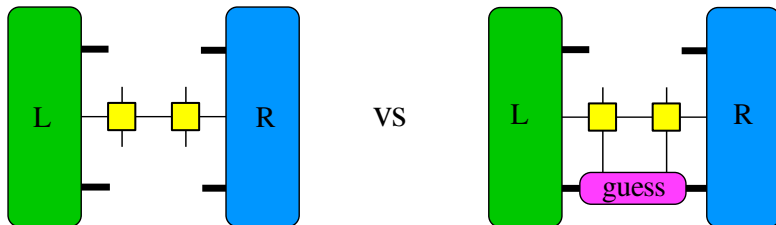
Lanczos algorithm

1. Let $v_1 \in \mathbb{C}^n$ be an arbitrary vector with [Euclidean norm](#) 1.
2. Abbreviated initial iteration step:
 1. Let $w'_1 = Av_1$.
 2. Let $\alpha_1 = w'^*_1 v_1$.
 3. Let $w_1 = w'_1 - \alpha_1 v_1$.
3. For $j = 2, \dots, m$ do:
 1. Let $\beta_j = \|w_{j-1}\|$ (also [Euclidean norm](#)).
 2. If $\beta_j \neq 0$, then let $v_j = w_{j-1}/\beta_j$,
 else pick as v_j an arbitrary vector with Euclidean norm 1 that is orthogonal to all of $v_1, \dots, v_{j-1}$.
 3. Let $w'_j = Av_j$.
 4. Let $\alpha_j = w'^*_j v_j$.
 5. Let $w_j = w'_j - \alpha_j v_j - \beta_j v_{j-1}$.

4. Let V be the matrix with columns $v_1, \dots, v_m$. Let $T =$

$$\begin{pmatrix} \alpha_1 & \beta_2 & & & & 0 \\ \beta_2 & \alpha_2 & \beta_3 & & & \\ & \beta_3 & \alpha_3 & \ddots & & \\ & & \ddots & \ddots & \beta_{m-1} & \\ & & & \beta_{m-1} & \alpha_{m-1} & \beta_m \\ 0 & & & & \beta_m & \alpha_m \end{pmatrix}.$$

Lanczos in MPS



- H_{eff} is too expensive
- In Lanczos: $H_{\text{eff}} \cdot v_{\text{guess}}$ - much cheaper! (see tutorials)

Control parameters

Control parameters

- Bond dimension
- Truncation weight
- Number of sweeps
- Number of Lanczos iterations

Control parameters

- Bond dimension
- Truncation weight
- Number of sweeps
- Number of Lanczos iterations

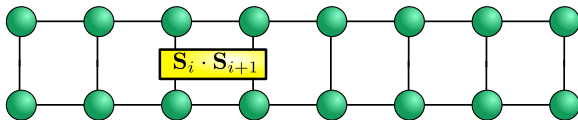
Two main DMRG practices:

- A few Lanczos iterations (≈ 5) + Many sweeps (≈ 100); D increases after the convergence is reached
- D increases every sweep, many Lanczos iterations (≈ 100), a few sweeps (≈ 5).

Observables

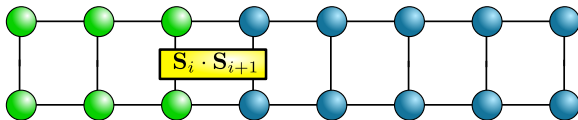
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



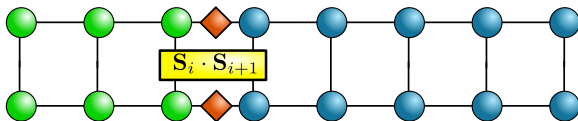
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



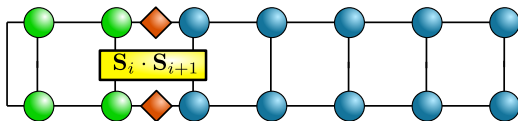
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



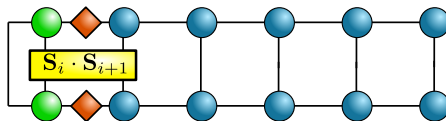
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



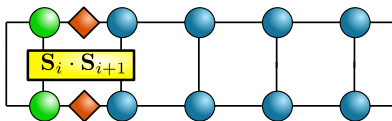
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



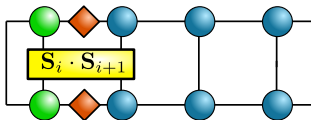
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



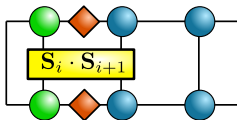
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



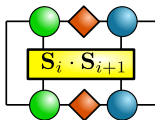
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



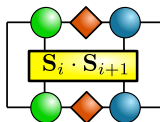
Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$

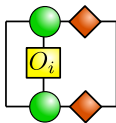


Observables

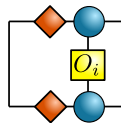
Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



On-site measures $\langle \Psi | O_i | \Psi \rangle$

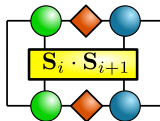


or

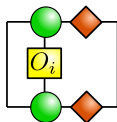


Observables

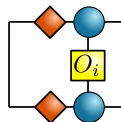
Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



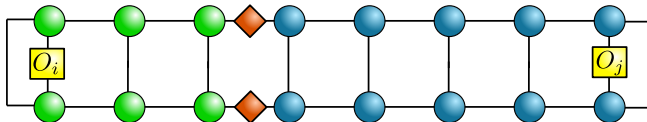
On-site measures $\langle \Psi | O_i | \Psi \rangle$



or



Long range correlations $\langle \Psi | O_i \cdot O_j | \Psi \rangle$



Entanglement

Entanglement entropy

$$S_{A|B} = -\text{Tr} \rho_A \log \rho_A = -\sum_{a=1}^D s_a^2 \log s_a^2 \quad (6)$$

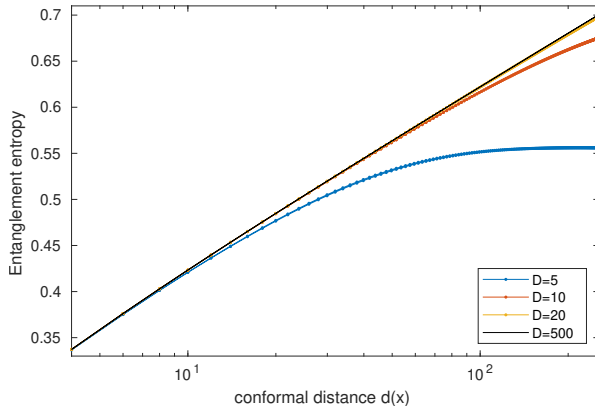


- DMRG is a low entangled approximation of a quantum state

Entanglement entropy



- DMRG is a low entangled approximation of a quantum state



Entanglement entropy

$$S_{A|B} = -\text{Tr} \rho_A \log \rho_A = -\sum_{a=1}^D s_a^2 \log s_a^2 \quad (7)$$

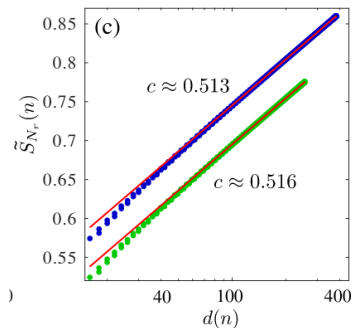
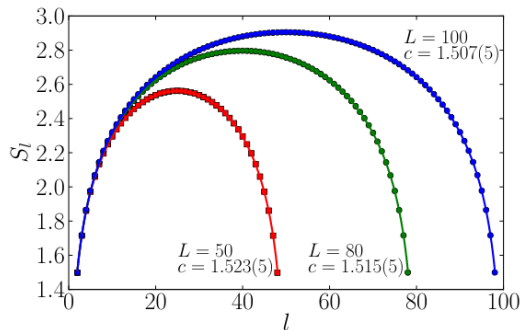


- DMRG is a low entangled approximation of a quantum state
- But the entanglement can itself be an observable

Central charge



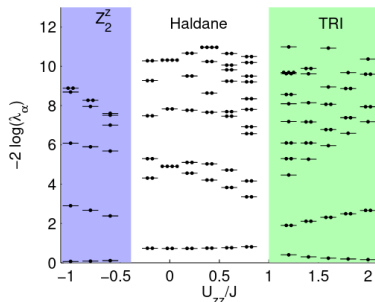
Calabrese-Cardy formula: $S_{\text{PBC}} = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi l}{L} \right) + c'_1$



Michaud et al. Phys. Rev. Lett. 108, 127202 (2012)

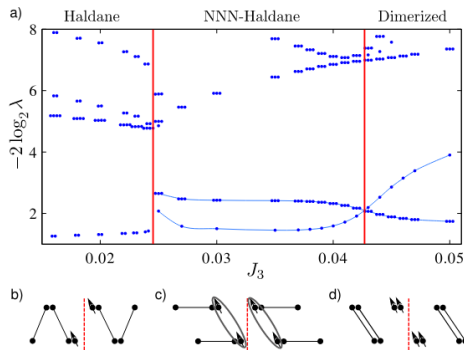
Entanglement spectra

$$\epsilon_a = -\log_2 s_a^2$$



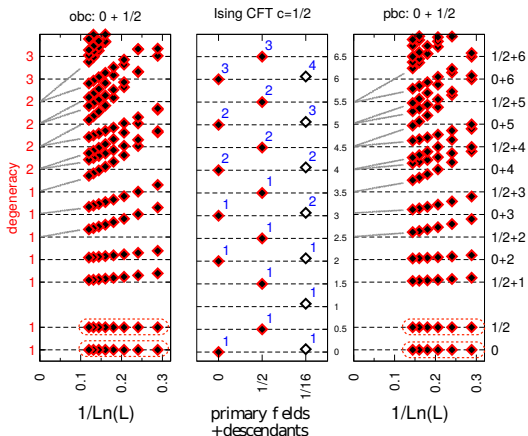
Pollmann et al., Phys. Rev. B 81, 064439 (2010)

- Identify topologically non-trivial phases
- Locate quantum phase transitions



Conformal towers from entanglement spectrum

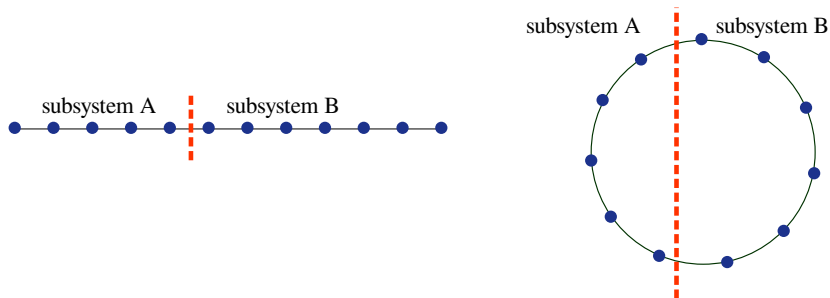
Transverse field Ising model



A.Läuchli, arxiv:1303.0741

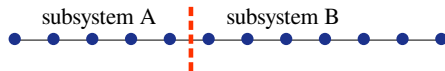
Boundary conditions

Periodic vs open boundary conditions

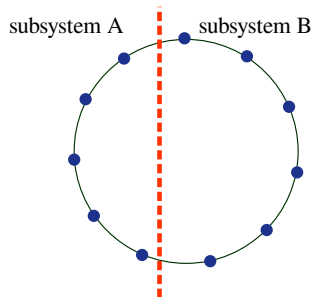


For periodic chains the entanglement is usually higher than for open ones

Periodic vs open boundary conditions



For periodic chains the entanglement is usually higher than for open ones

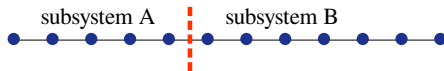


Calabrese-Cardy formulas for critical systems:

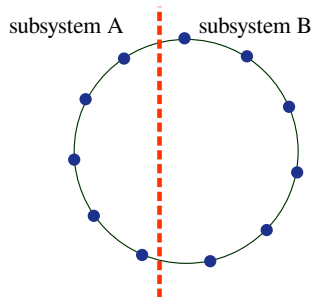
$$S_{\text{PBC}} = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi l}{L} \right) + c'_1, \quad (8)$$

$$S_{\text{OBC}} = \frac{c}{6} \log \left(\frac{2L}{\pi a} \sin \frac{\pi l}{L} \right) + c'_1/2 + \log \tilde{g}. \quad (9)$$

Periodic vs open boundary conditions



For periodic chains the entanglement is usually higher than for open ones

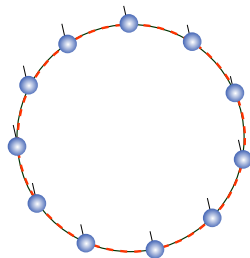
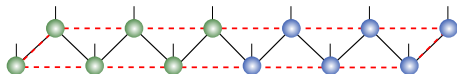
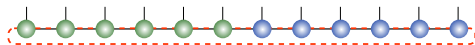


Calabrese-Cardy formulas for critical systems.

$$S_{\text{PBC}} = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi l}{L} \right) + c'_1, \quad (10)$$

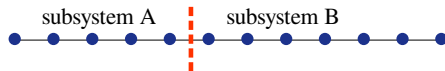
$$S_{\text{OBC}} = \frac{c}{6} \log \left(\frac{2L}{\pi a} \sin \frac{\pi l}{L} \right) + c'_1/2 + \log \tilde{g}. \quad (11)$$

PBC with MPS

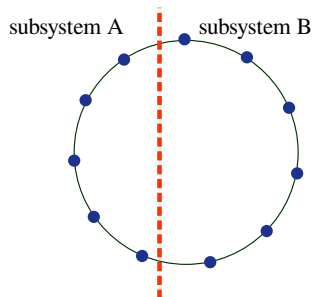


red = lattice; black = tensor network

Periodic vs open boundary conditions



For periodic chains the entanglement is usually higher than for open ones

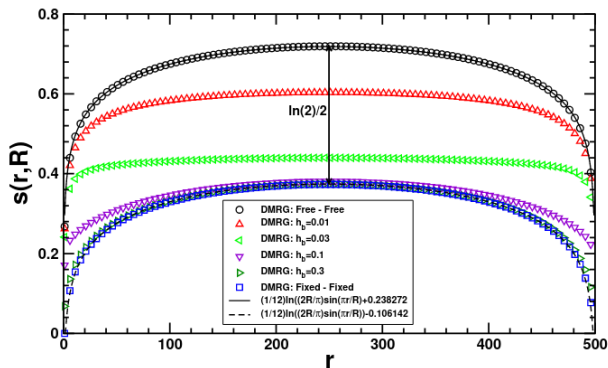


Calabrese-Cardy formulas for critical systems. **Note: the boundary term $\tilde{g}$**

$$S_{\text{PBC}} = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi l}{L} \right) + c'_1, \quad (12)$$

$$S_{\text{OBC}} = \frac{c}{6} \log \left(\frac{2L}{\pi a} \sin \frac{\pi l}{L} \right) + c'_1/2 + \log \tilde{g} \quad (13)$$

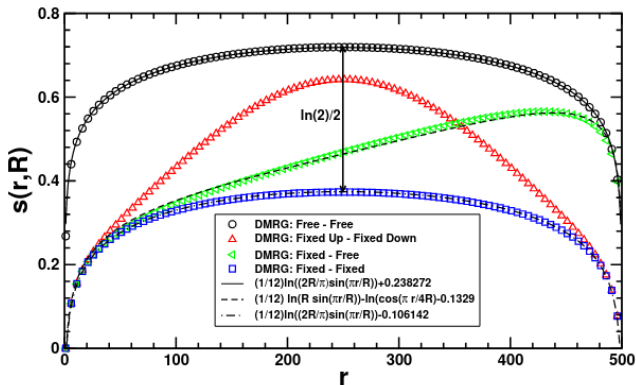
Fixed BC = Lower entanglement



Affleck, Laflorencie, Sorensen, J. Phys. A: Math. Theor. 42 504009 (2009)

Sometimes boundary conditions are unintentionally "fixed" by the model (e.g. Heisenberg chain)

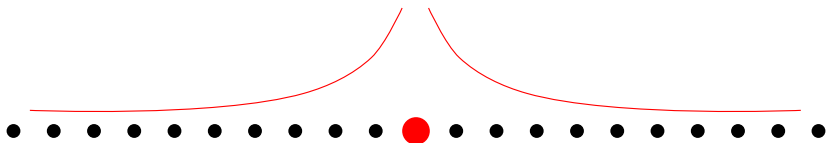
Non-symmetric BC = different scaling!



Affleck, Laflorencie, Sorensen, J. Phys. A: Math. Theor. 42 504009 (2009)

Friedel oscillations

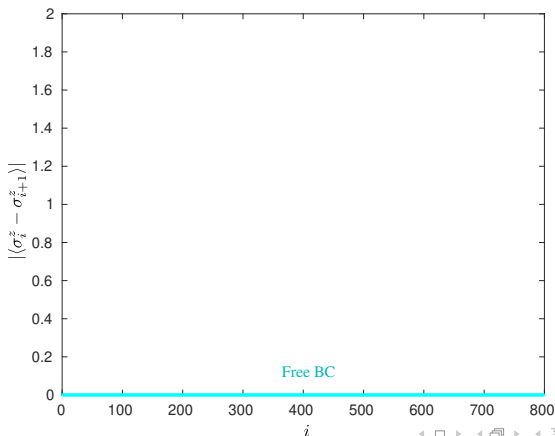
- Response of the system to an impurity
- In the gapped phase it decays exponentially
- At the critical point - with the corresponding critical exponent
- Open boundary conditions = impurity
- Prediction by boundary-CFT



Broken symmetry

Transverse field Ising model:

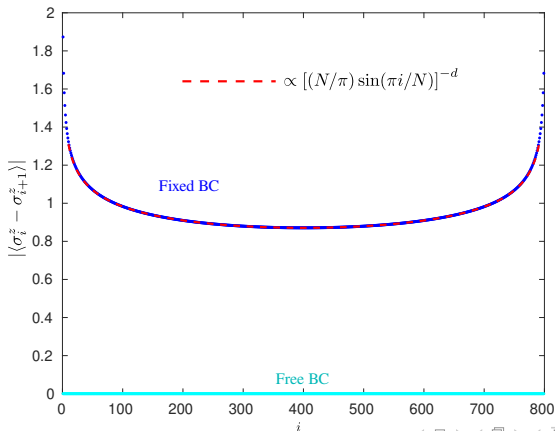
$$H = \sum_i \sigma_i^z \sigma_{i+1}^z + h \sigma_i^x$$



Fixed BC = Friedel oscillations

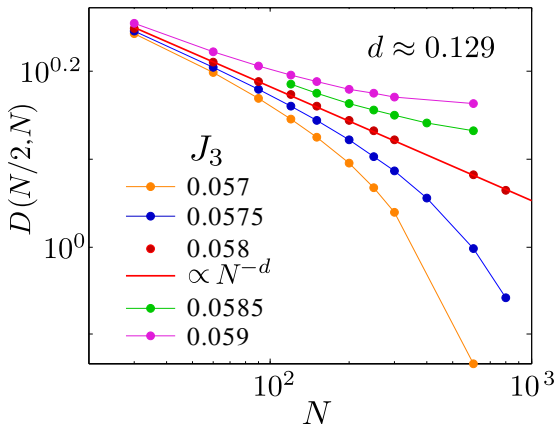
Transverse field Ising model:

$$H = \sum_i \sigma_i^z \sigma_{i+1}^z + h \sigma_i^x$$



Locate phase transitions

$$H = \sum_i \sigma_i^z \sigma_{i+1}^z + h \sigma_i^x$$



By-path in computing correlations

Distant correlations

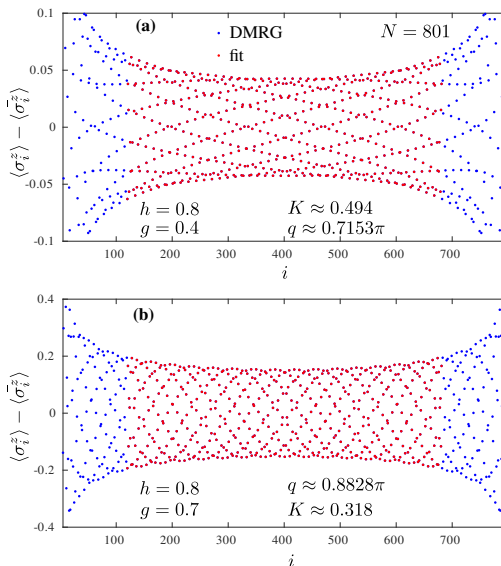
$$\langle O_i O_{i+r} \rangle - \langle O_i \rangle \langle O_{i+r} \rangle \propto r^\eta$$



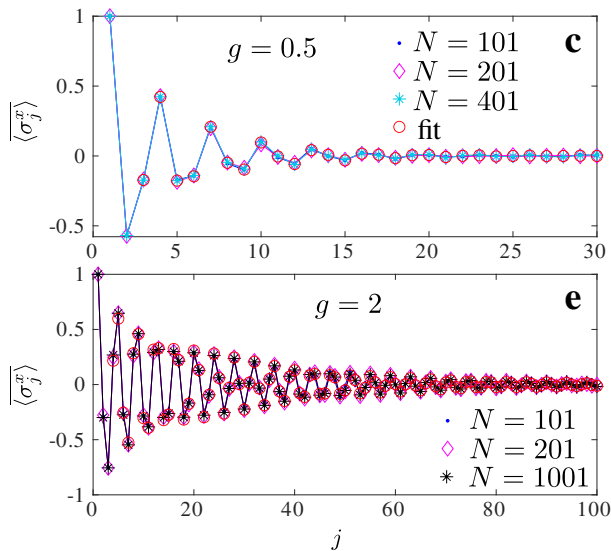
Friedel oscillations

$$\langle O_r \rangle \propto r^{\eta/2}$$

Extract Luttinger liquid exponents

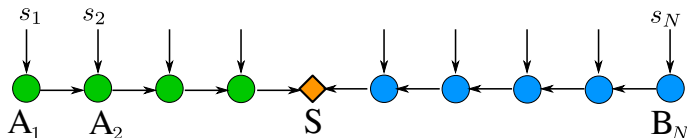


Way more accurate in case of disorder and noise



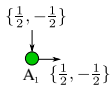
Abelian symmetries

Abelian symmetry



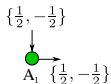
- Assign quantum numbers - labels to physical bonds of MPS
- Using fusion rules of the symmetry, find quantum numbers on auxiliary legs
- When local basis is sorted according to the quantum number of states, the MPS takes a block-diagonal form

Abelian symmetry. Examples

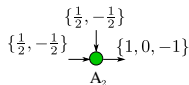


$$-\frac{1}{2} \begin{array}{c} \frac{1}{2} - \frac{1}{2} M_1 \\ \sigma_1 \end{array}$$

Abelian symmetry. Examples

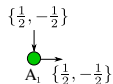


$$\begin{array}{c} \frac{1}{2} \quad -\frac{1}{2} \quad M_1 \\ \frac{1}{2} \\ -\frac{1}{2} \\ \sigma_1 \end{array} \begin{array}{|c|c|} \hline \text{gray} & \text{white} \\ \hline \text{white} & \text{gray} \\ \hline \end{array}$$

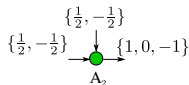


$$\begin{array}{c} 1 \quad 0 \quad -1 \quad M_2 \\ 1/2 \otimes 1/2 \\ 1/2 \otimes -1/2 \\ -1/2 \otimes 1/2 \\ -1/2 \otimes -1/2 \\ M_1 \otimes \sigma_2 \end{array} \begin{array}{|c|c|c|} \hline \text{gray} & \text{white} & \text{white} \\ \hline \text{white} & \text{gray} & \text{gray} \\ \hline \text{white} & \text{gray} & \text{gray} \\ \hline \text{white} & \text{white} & \text{gray} \\ \hline \end{array}$$

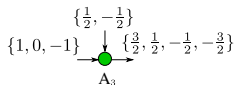
Abelian symmetry. Examples

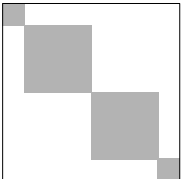


$$\begin{array}{c} \frac{1}{2} \quad -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \sigma_1 \end{array} M_1$$

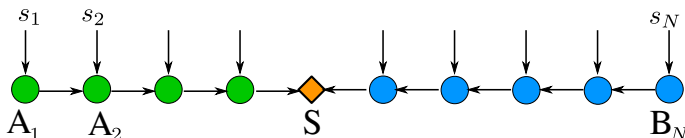



$$\begin{array}{c} 1 \quad 0 \quad -1 \\ 1/2 \otimes 1/2 \\ 1/2 \otimes -1/2 \\ -1/2 \otimes 1/2 \\ -1/2 \otimes -1/2 \\ M_1 \otimes \sigma_2 \end{array} M_2$$

$$\begin{array}{c} \frac{3}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{3}{2} \\ M_2 \otimes \sigma_3 \end{array} M_3$$


Abelian symmetry



- Assign quantum numbers - labels to physical bonds of MPS
- Using fusion rules of the symmetry, find quantum numbers on auxiliary legs
- When local basis is sorted according to the quantum number of states, the MPS takes a block-diagonal form
- Select the right symmetry sector by matching the right combination of quantum numbers

Not implemented symmetries

Keep your guess in the right sector

Constraints

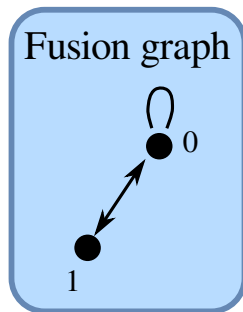
Nearest-neighbor blockade: $n_i n_{i+1} = 0$

Fusion graph for $r = 1$ Rydberg blockade

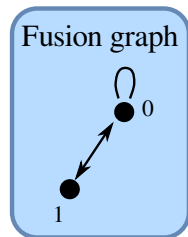
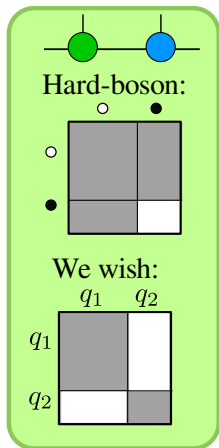
$$\dots \bigcirc + \bullet = \dots \bigcirc \bullet$$

$$\dots \bullet + \bigcirc = \dots \bullet \bigcirc$$

$$\dots \bigcirc + \bigcirc = \dots \bigcirc \bigcirc$$

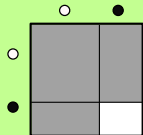


Encoding the $r = 1$ Rydberg blockade

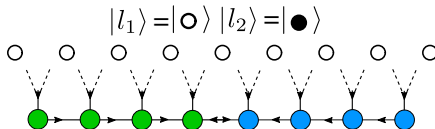
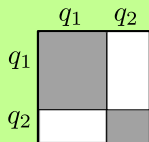


Encoding the $r = 1$ Rydberg blockade

Hard-boson:



We wish:

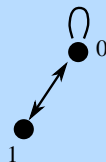


New local Hilbert space:

$$|h_1\rangle = |\circ\circ\rangle \quad |h_2\rangle = |\circ\bullet\rangle \quad |h_3\rangle = |\bullet\circ\rangle$$

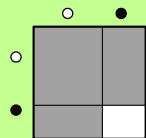
Span local degrees
of freedom over
two sites

Fusion graph

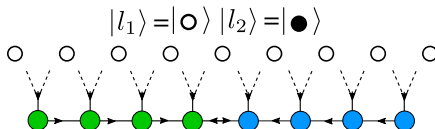
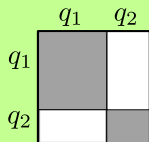


Encoding the $r = 1$ Rydberg blockade

Hard-boson:



We wish:



New local Hilbert space:

$$|h_1\rangle = |\text{white dot white dot}\rangle \quad |h_2\rangle = |\text{white dot black dot}\rangle \quad |h_3\rangle = |\text{black dot white dot}\rangle$$

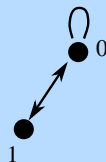
Span local degrees
of freedom over
two sites

For each neighbors there is a
common boson site:

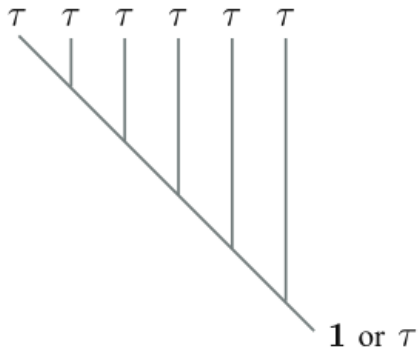


use its states as **quantum labels!**

Fusion graph



Fusion graph for Fibonacci anyons



$$I \times \tau = \tau$$

$$\tau \times \tau = I + \tau$$

Trebst et al., Prog. Theor. Phys. Supp. 176, 384 (2008)

Fusion graph for Fibonacci anyons

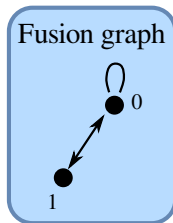
$$I \times \tau = \tau$$

$$\tau \times \tau = I + \tau$$

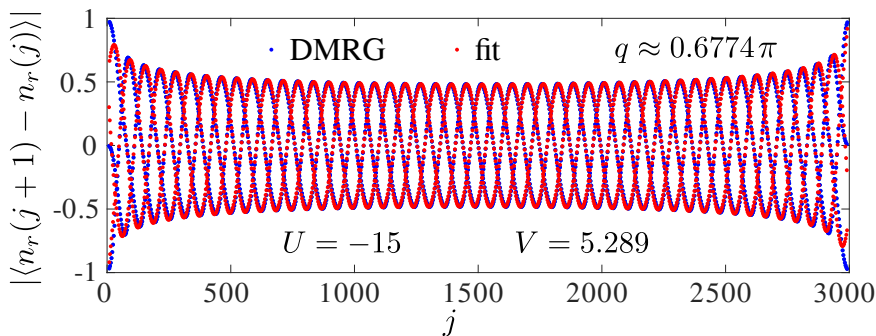
$$\dots \bigcirc + \bullet = \dots \bigcirc \bullet$$

$$\dots \bullet + \bigcirc = \dots \bullet \bigcirc$$

$$\dots \bigcirc + \bigcirc = \dots \bigcirc \bigcirc$$



Why constrained TN?



- Convergence for critical systems with many thousands of sites
- Wide range of applications: Rydberg blockade, anyons, SUSY models and topological data analysis,...

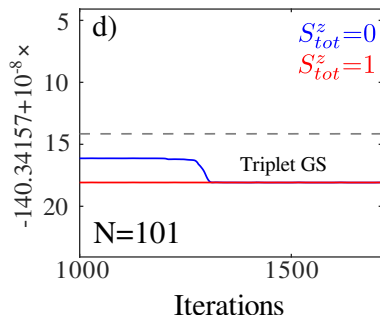
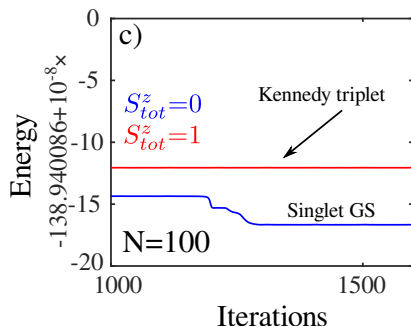
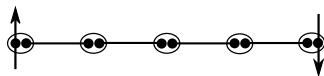
Excitation spectrum

Excitation spectrum with DMRG/MPS

- ① The excited state is the 'ground-state' of the different symmetry sector

Example: Kennedy triplet in Haldane chain

$$H = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$



Excitation spectrum with DMRG/MPS

- ① The excited state is the 'ground-state' of the different symmetry sector
- ② Conventional DMRG: Mixed states
 - The ground-state is spoiled
 - Heavy memory usage

Excitation spectrum with DMRG/MPS

- ① The excited state is the 'ground-state' of the different symmetry sector
- ② Conventional DMRG: Mixed states
 - No longer variational
 - Heavy memory usage
- ③ MPS: Construct the lowest-energy state orthogonal to the previously constructed ones
 - Time consuming
 - Accumulation of the error

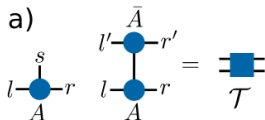
Excitation spectrum with DMRG/MPS

- ① The excited state is the 'ground-state' of the different symmetry sector
- ② Conventional DMRG: Mixed states
 - No longer variational
 - Heavy memory usage
- ③ MPS: Construct the lowest-energy state orthogonal to the previously constructed ones
 - Time consuming
 - Accumulation of the error
- ④ Elementary excitations + plane-wave superposition
 - Translation invariant MPS
 - Not suitable for OBC

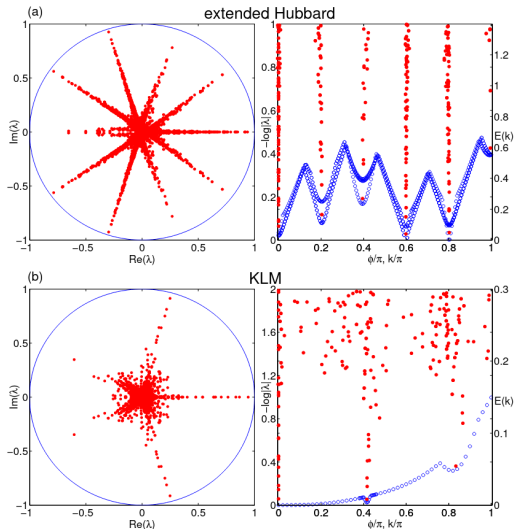
$$|\Phi_p^k(B)\rangle = \sum_n e^{ipn} \begin{array}{c} \text{---} \boxed{A} \text{---}^j \boxed{A} \text{---}^j \boxed{B} \text{---}^j \boxed{A} \text{---}^j \boxed{A} \text{---} \\ \text{\scriptsize } s_{n-2} \quad s_{n-1} \quad s_n \quad s_{n+1} \quad s_{n+2} \end{array}$$

k

Transfer matrix



Zauner et al, New J. Phys. 17 (2015)
053002



Excitation spectrum with DMRG/MPS

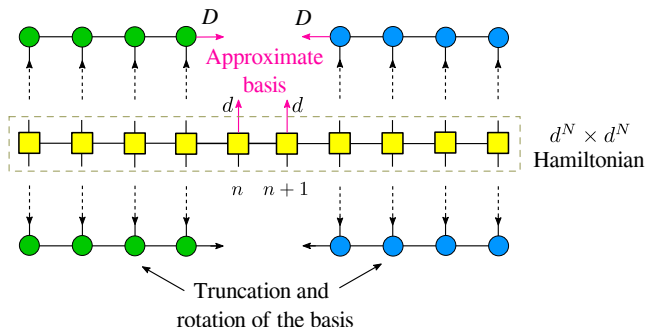
- ① The excited state is the 'GS' of the different symmetry sector
- ② Conventional DMRG: Mixed states
 - No longer variational & Heavy memory usage
- ③ MPS: Construct the lowest-energy state orthogonal to the previously constructed ones
 - Time consuming & Accumulation of the error
- ④ MPS: Domain wall/ special tensor/ transfer matrix
 - Translation invariant MPS

There is a cheaper option:

Sometimes it is sufficient to target multiple eigenstates of the effective Hamiltonian and keep track of the energies as a function of iterations

[NC, Mila, Phys.Rev.B **96**, 054425 (2017)]

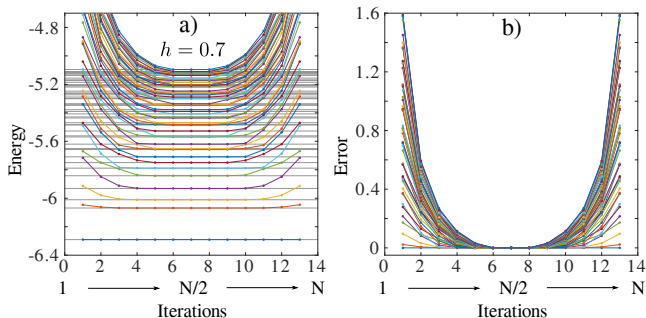
Effective Hamiltonian



- The Hamiltonian is written in a truncated and rotated basis.
- This basis is selected for the ground state.
- Could this basis be suitable for other low-energy states?

Trivial case - non-truncated MPS

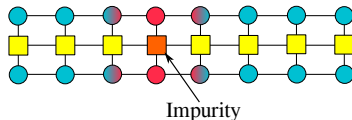
When no truncation is imposed and **all** basis states are kept in MPS, the DMRG is equivalent to exact diagonalization and one can access the entire spectrum!



When does it work?

Local impurities

- Localized excitations
- MPS is the same except for a few sites



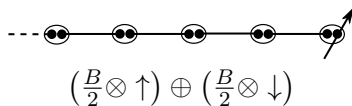
When does it work?

Edge states

- Edge spins are entangled through the entire network
- All edge states are in the basis

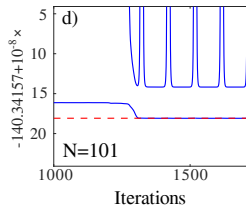
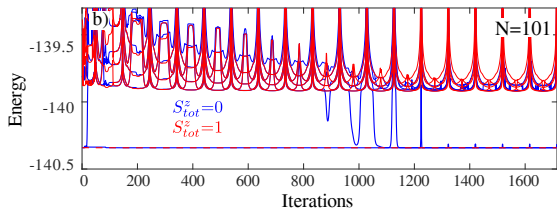
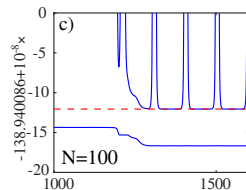
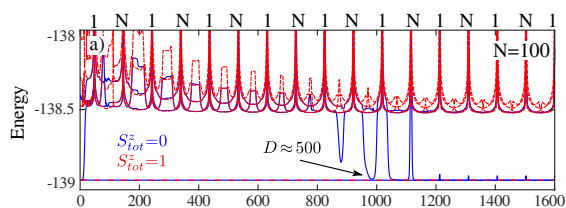
Local impurities

- Localized excitations
- MPS is the same except for a few sites

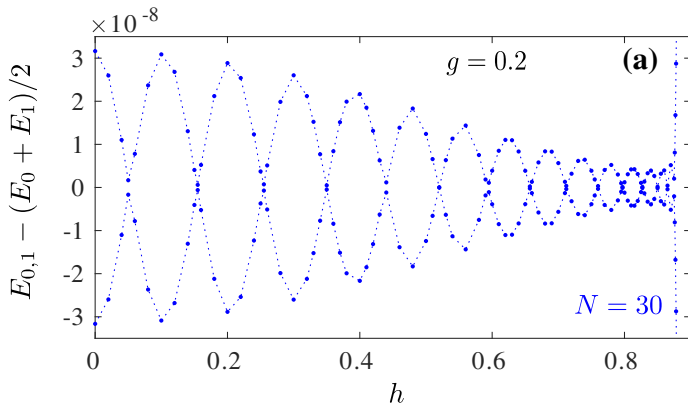


Edge states in the Haldane chain

$$H = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$



Exact zero modes in Majorana chain



When correlations are incommensurate the effective coupling between edge states can be continuously tuned $J_{\text{eff}} \propto \cos(q \cdot L)$

Zero modes appear and we can capture them with nearly machine precision!

When does it work?

Critical systems

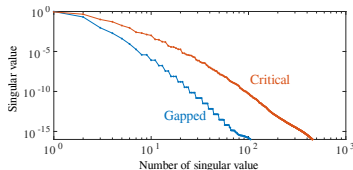
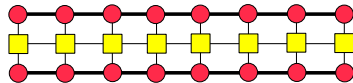
- Divergent correlation length
- Slow decay of Schmidt values
- Special structure of spectrum

Edge states

- Edge spins are entangled through the entire network
- All edge states are in the basis

Local impurities

- Localized excitations
- MPS is the same except for a few sites

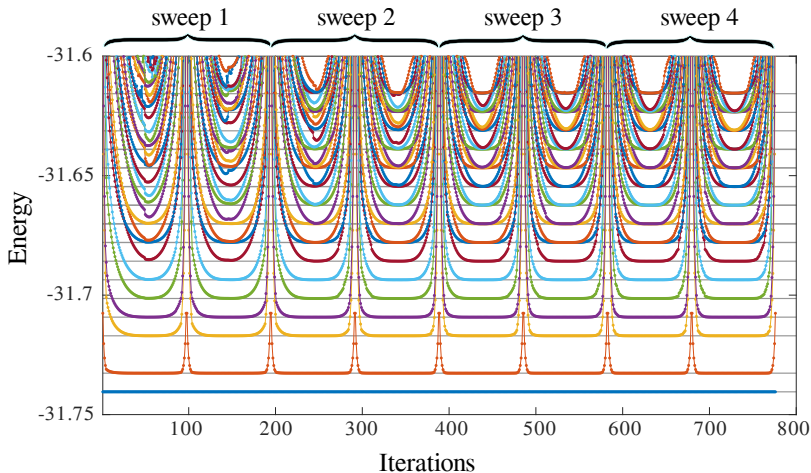


Transverse field Ising model

$$H = \sum_i J S_i^x S_{i+1}^x + h S_i^z$$

- Critical at $h = J/2$
- Solved by Jordan-Wigner transformation
- Corresponds to the minimal model (4,3) in CFT

Transverse field Ising model. Excitation spectrum

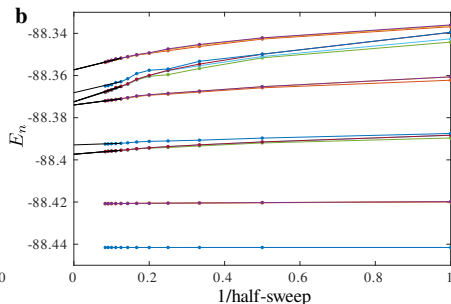
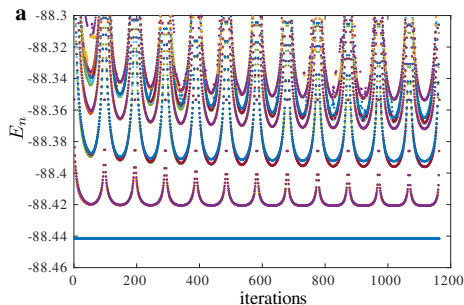


- 30 states within a single run!
- Flat modes signal convergence

NC, F. Mila, Phys. Rev. B 96, 054425'17

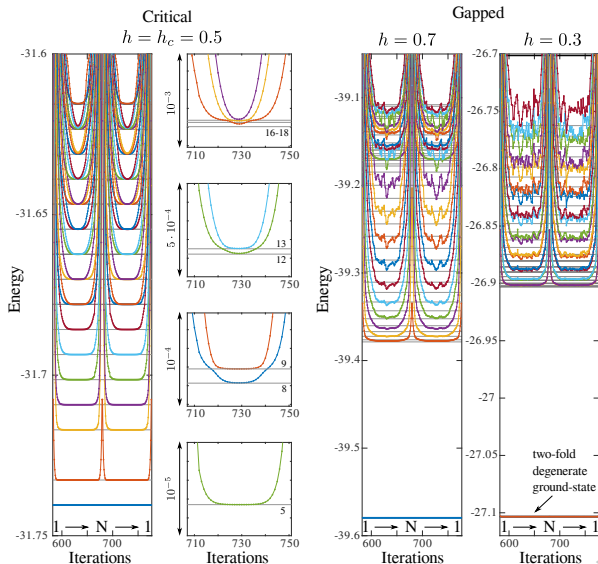
The convergence is sometimes tricky

4-state Potts with free BC:



NC, SciPost Phys. Core 5, 031 (2022)

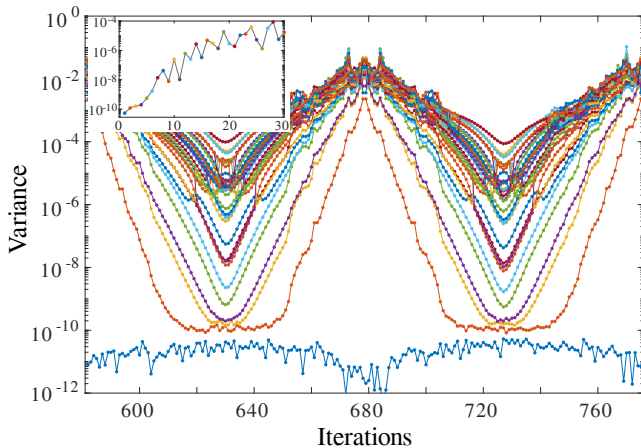
Transverse field Ising model. Excitation spectrum



- Remarkable accuracy for critical system
- Wrong spectrum for gapped system
- promising results for multi-site DMRG
Banuls et al, in prep

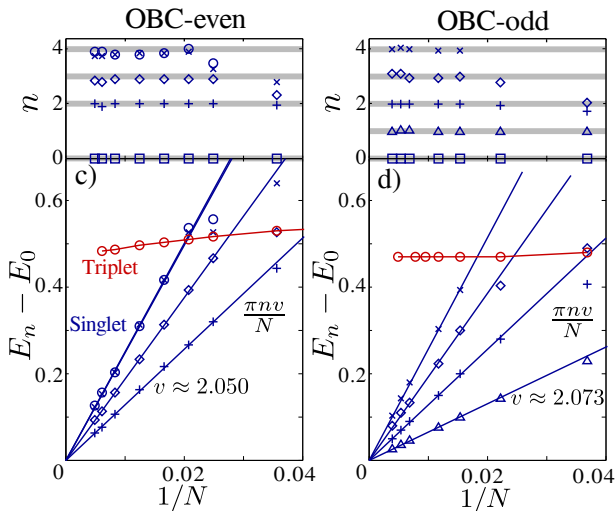
NC, Mila, PRB 2017

States are also good!



NC, Mila, PRB 2017

Non-magnetic Ising transition in a spin-1 chain



- Singlet-triplet gap is **open**
- Critical scaling of the gap in the singlet sector
- **N even**
 I conformal tower
- **N odd**
 ϵ conformal tower

NC, Affleck, Mila, PRB **93**,
241108'16

There is a reason why it works for CFT

Extracting the Speed of Light from Matrix Product States

Phys. Rev. Lett. 131, 226502 (2023)

Alexander A. Eberharter,¹ Laurens Vanderstraeten,² Frank Verstraete,² and Andreas M. Läuchli^{3,4}

¹Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria

²Department of Physics and Astronomy, University of Ghent, Belgium

³Laboratory for Theoretical and Computational Physics, Paul Scherrer Institute, 5232 Villigen, Switzerland

⁴Institute of Physics, École Polytechnique Fédérale de Lausanne (EPFL), 1015 Lausanne, Switzerland

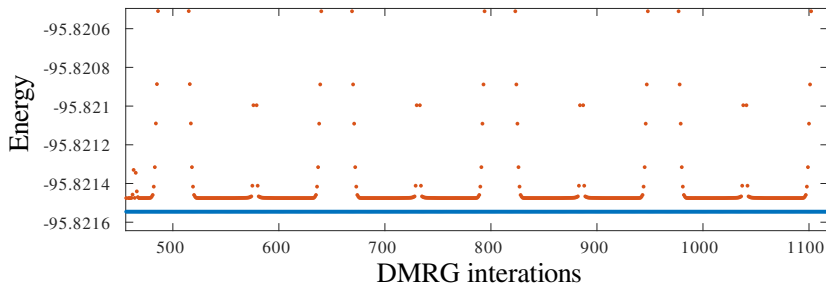
(Dated: March 2, 2023)

We provide evidence that the spectrum of the local effective Hamiltonian and the transfer operator in infinite-system matrix product state simulations are identical up to a global rescaling factor, i.e. the speed of light of the system, when the underlying system is described by a 1+1 dimensional CFT. We provide arguments for this correspondence based on a path integral point of view. This observation turns out to yield very precise estimates for the speed of light in practice, confirming exact results to high precision where available, but also allowing us to finally determine the speed of light of the non-integrable, critical $SU(2)$ Heisenberg chains with half-integer spin $S > 1/2$

But it also works for non-CFT

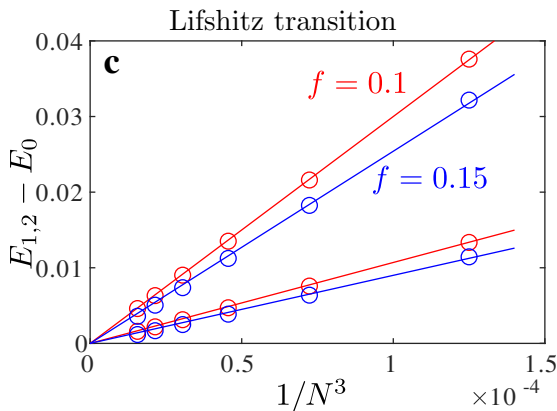
Infinite randomness criticality

$$H = \sum J_i S_i^x S_{i+1}^x + \sum h_i S_i^z$$



accuracy about 10^{-8} , largely controlled by the truncation error

But it also works for non-CFT



Dynamical critical exponent $z = 3$.

Time evolution

Time evolution: real and imaginary

The goal is to compute action operators:

$$e^{-iHt} \quad \text{and} \quad e^{-\beta H}$$

First-order Trotter decomposition

$$e^{-i\hat{H}\tau} = e^{-i\hat{h}_1\tau} e^{-i\hat{h}_2\tau} e^{-i\hat{h}_3\tau} \dots e^{-i\hat{h}_{L-3}\tau} e^{-i\hat{h}_{L-2}\tau} e^{-i\hat{h}_{L-1}\tau} + O(\tau^2),$$

Second-order Trotter decomposition - better accuracy at no cost

$$e^{-i\hat{H}\tau} = e^{-i\hat{H}_{\text{odd}}\tau/2} e^{-i\hat{H}_{\text{even}}\tau} e^{-i\hat{H}_{\text{odd}}\tau/2} + O(\tau^3),$$

Time evolution: real and imaginary

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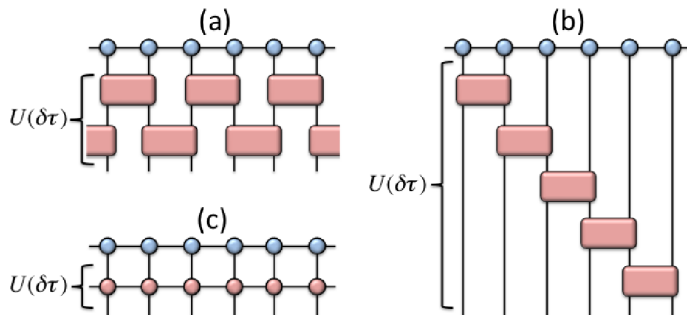
For very long i-times quantum state approaches the ground-state:

$$|E_0\rangle = \lim_{\tau \rightarrow \infty} \frac{e^{-\tau H} |\Psi(0)\rangle}{\sqrt{|\langle \Psi(\tau) | \Psi(\tau) \rangle|}} \quad |\Psi(\tau)\rangle = e^{-\tau H} |\Psi(0)\rangle$$

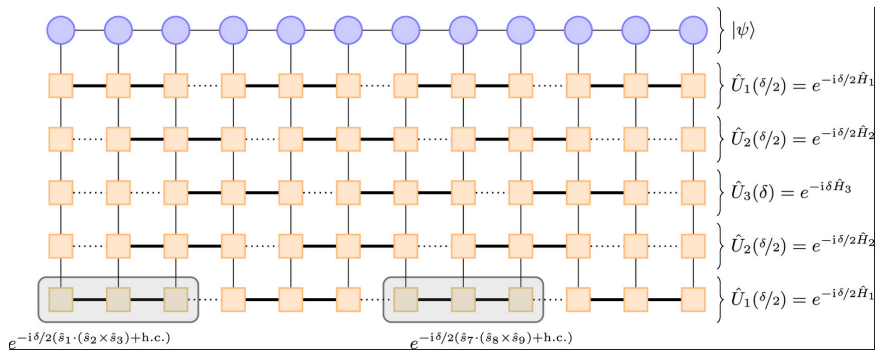
Time evolution: various ways to apply

Second-order Trotter decomposition - better accuracy at no cost

$$e^{-i\hat{H}\tau} = e^{-i\hat{H}_{\text{odd}}\tau/2} e^{-i\hat{H}_{\text{even}}\tau} e^{-i\hat{H}_{\text{odd}}\tau/2} + O(\tau^3),$$

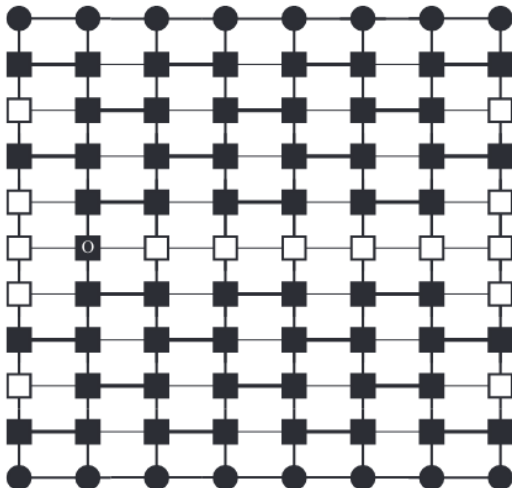


Time evolution: longer-range terms



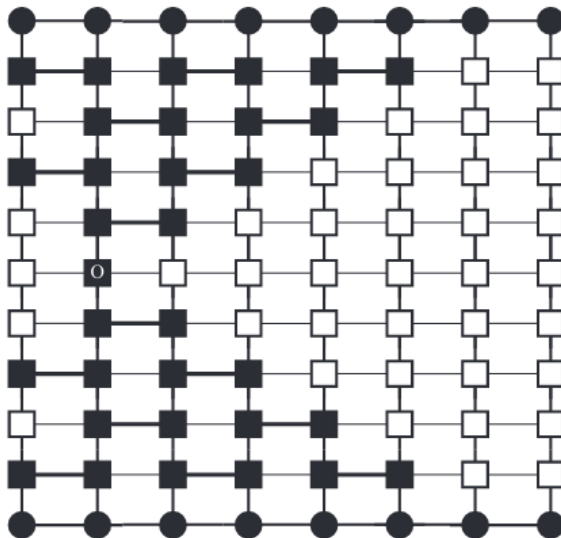
One can also use swap-gates, but keep an eye on a complexity!
For truly long-range order - TDVP

Extracting observables

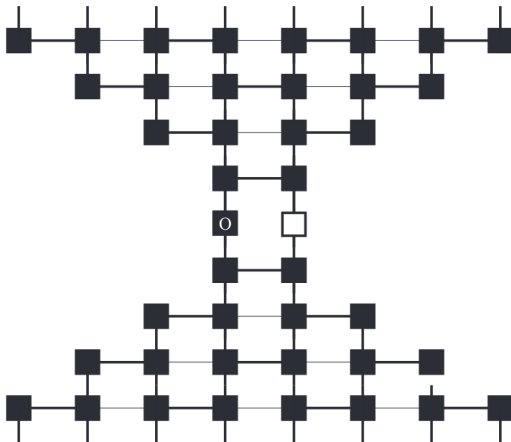


Schollwoeck, Annals of Physics 326, 96 (2011)

Extracting observables



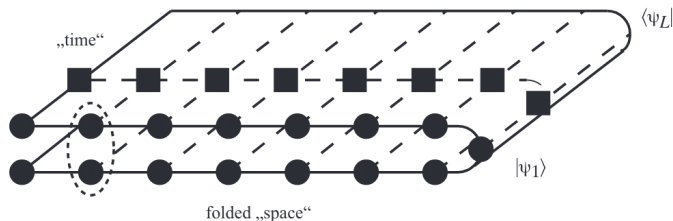
Light cone



Schollwoeck, Annals of Physics 326, 96 (2011)

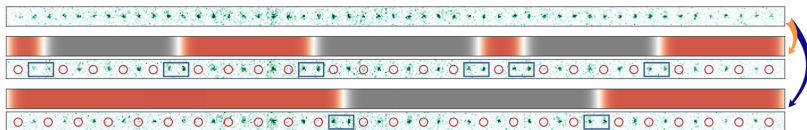
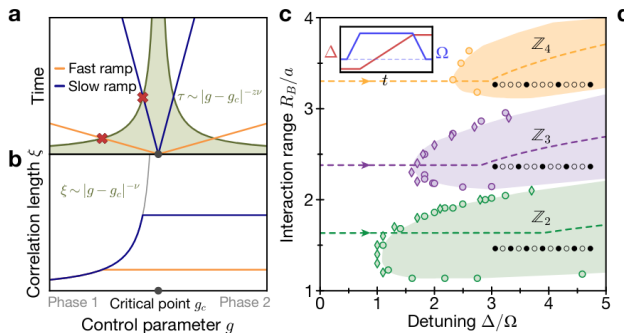
Folding

folding of forward and backward timesteps leads to some cancellations
in entanglement



Schollwoeck, Annals of Physics 326, 96 (2011)

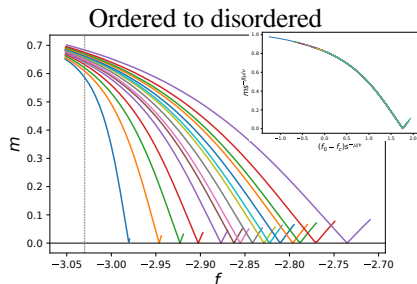
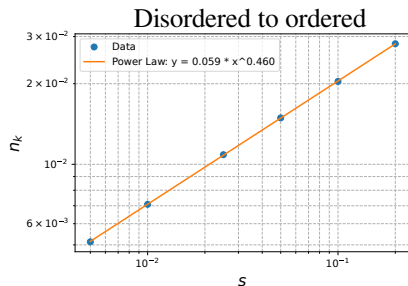
Motivation: Rydberg experiments



Keesling et al., Nature'19

KZ: the faster you sweep the more domain walls you end up with

Kibble-Zurek and Finite-time scaling



Kibble-Zurek mechanism:

The faster you sweep the more domain walls you end up with

Finite-time scaling:

The slower you sweep the latter the order parameter will vanish

Kibble-Zurek mechanism in space

DMRG and area law

Low energy states (local H)



Area law:

Ground states of local Hamiltonians are less entangled than a random state in the Hilbert space

Number of relevant states (gapped): $D \propto \exp(S)$

GS of local Hamiltonians

Area law: $S_A(L) \propto L^{d-1}$

1D: $S_A(L) = \text{const}$

2D: $S_A(L) \propto L$

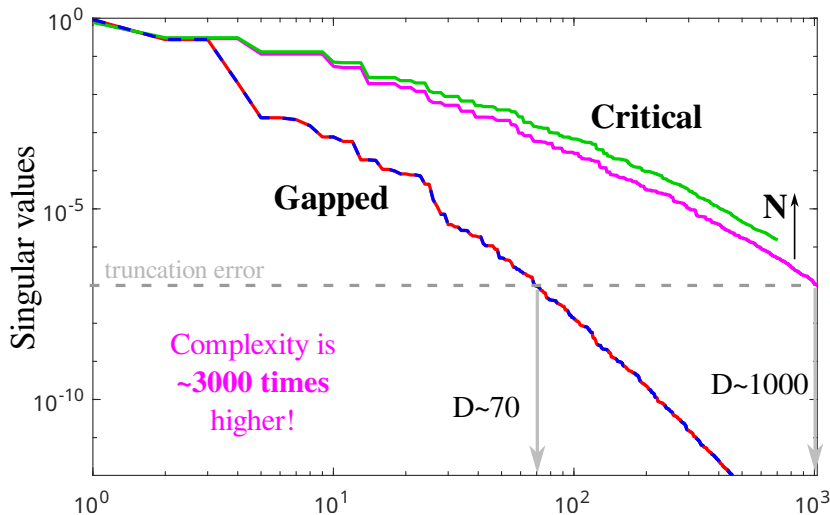
Random state

Volume law: $S_A(L) \propto L^d$

Critical state in 1D

$S_A(L) \propto \log(L)$

Singular values decay much slower...



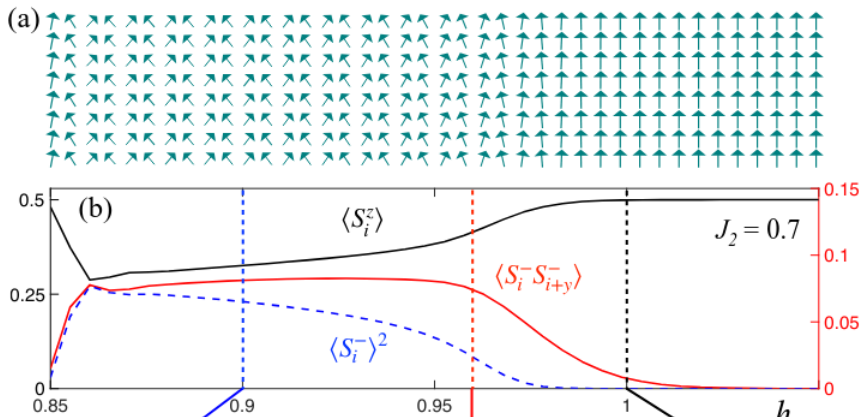
... and computational costs increase correspondingly

Scan-DMRG

Cuts through the phase diagram

Long-cylinder scans (aka Steve White's trick)

One can get all phases in a single run



Jiang, Romhanyi, White, Zhitomirsky, Chernyshev, Phys. Rev. Lett. 130, 116701 (2023)

Long-cylinder scans (aka Steve White's trick)

One can get all phases in a single run

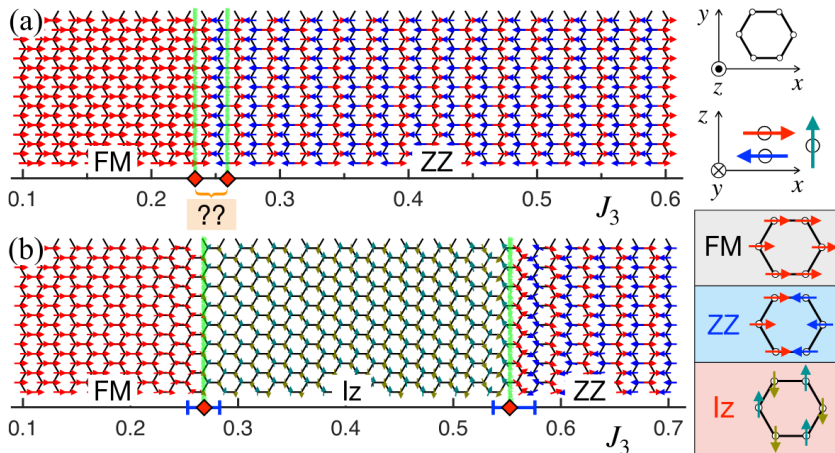


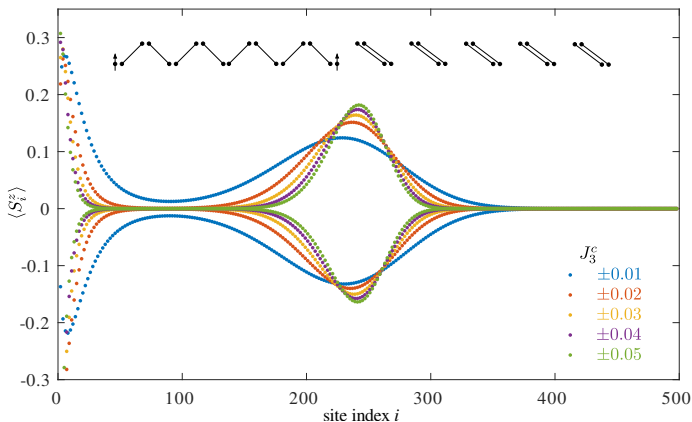
FIG. 2. Long-cylinder scans of the J_1 - J_3 model (1) vs J_3 in the (a) Heisenberg ($\Delta=1$) and (b) XY ($\Delta=0$) limit. The arrows
Jiang, White, Chernyshev, Phys. Rev. B 108, L180406 (2023)

Great tool to probe domain walls

... and to get the first insight into the nature of the transition

Domain walls with scan-DMRG

One can easily distinguish magnetic vs non-magnetic domain walls



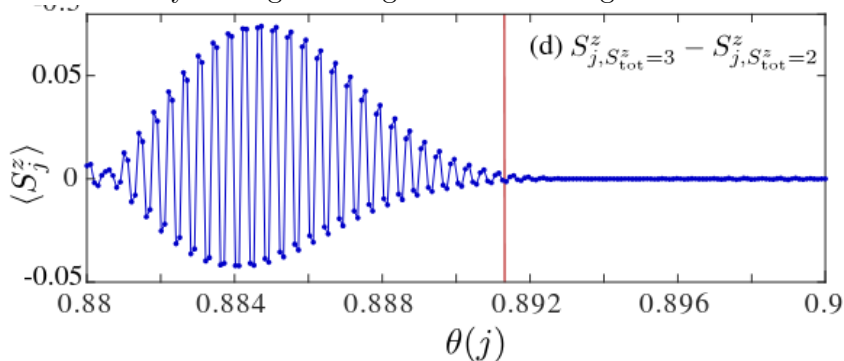
Magnetic domain wall between the Haldane and dimerized phases

→ *magnetic* WZW transition

NC, Phys. Rev. B 110, 144401 (2024)

Domain walls with scan-DMRG

One can easily distinguish magnetic vs non-magnetic domain walls



Magnetic excitations are spread over one of the domains.

Non-magnetic domain wall $\rightarrow$ *non-magnetic* Ising transition

Chepiga, Mila, Phys. Rev. B 100, 104426 (2019)

scan-DMRG can be a great tool to
identify quantum phase transitions

Scan-DMRG is a spatial version of Kibble-Zurek

Conventional Kibble-Zurek

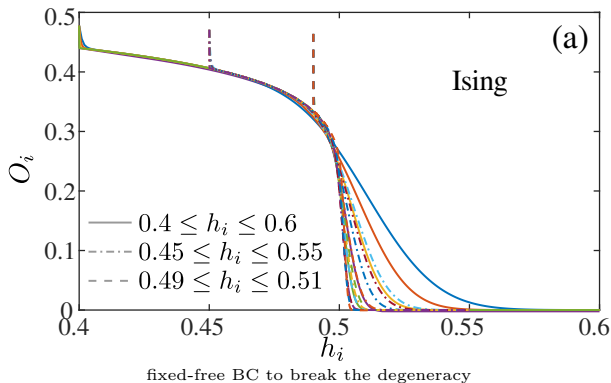
- Time step τ
- Characteristic time
 $t \sim \tau^\nu / (1+\nu)$
- Distance to the transitions
 $\epsilon \sim \frac{t}{\tau} \sim \tau^{\frac{-1}{1+\nu}}$
- Order parameter
 $O \propto \tau^{\frac{-\beta}{1+\nu}}$

Spatial Kibble-Zurek

- Gradient step $\delta = \frac{h_N - h_1}{N-1}$
- Characteristic length
 $\xi \sim (g - g_c)^{-\nu} \sim x^{-\nu} \delta^{\frac{-\nu}{1+\nu}}$
- Distance to the transition:
 $x \sim (g - g_c) \delta^{\frac{-1}{1+\nu}}$
- Order parameter
 $O \propto x^\beta \delta^{\frac{\beta}{1+\nu}}$

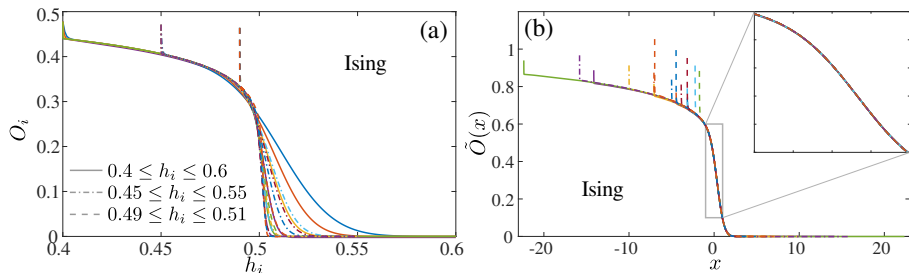
Ising chain with a gradient. Raw data

$$H = -J \sum_i S_i^x S_{i+1}^x - \sum_i h_i S_i^z; \quad h_i \in [h_{st}, h_{en}]$$



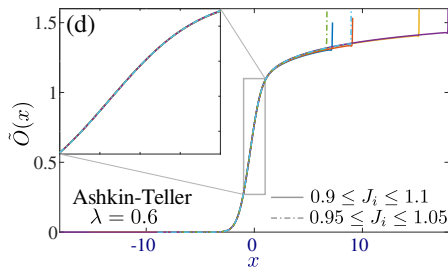
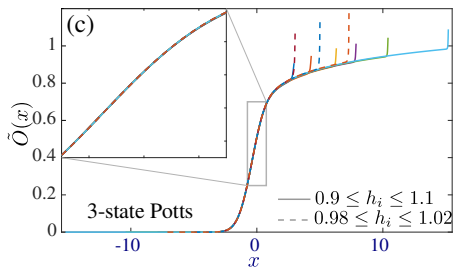
Ising chain with a gradient. Data collapse

$$H = -J \sum_i S_i^x S_{i+1}^x - \sum_i h_i S_i^z; \quad h_i \in [h_{st}, h_{en}]$$



NC, Phys. Rev. B 110, 144401 (2024)

3-state Potts and Ashkin-Teller chains with gradient

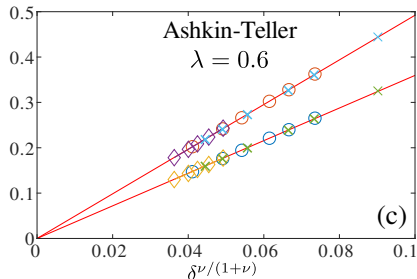
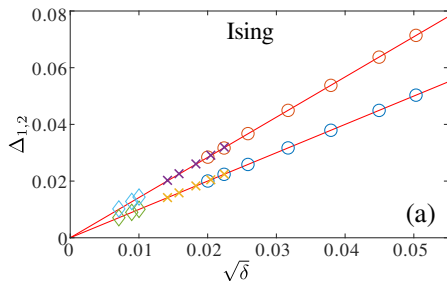


- Works equally well for field and interaction gradients
- 3-state Potts: $\nu = 5/6$ and $\beta = 1/9$
- Ashkin-Teller with $\lambda = 0.6 \rightarrow \nu \approx 0.7748$ and $\beta \approx 0.0969$

Energy gap

Universal scaling of the energy gap

$$\Delta \propto \xi^{-1} \Rightarrow \Delta \propto \delta^{\frac{\nu}{1+\nu}}$$



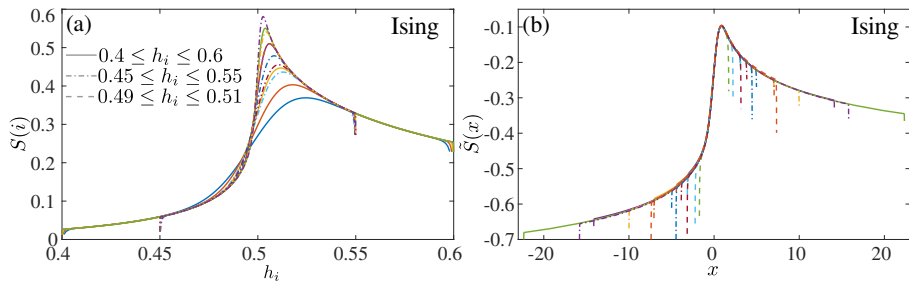
- Ising $\frac{\nu}{1+\nu} = 0.5$. Numerics: 0.4995
- Ashkin-Teller ($\lambda = 0.6$) $\frac{\nu}{1+\nu} \approx 0.4366$. Numerics: 0.4411 (within 1%)

NC, Phys. Rev. B 110, 144401 (2024)

Entanglement entropy

Entanglement entropy. Ising transition

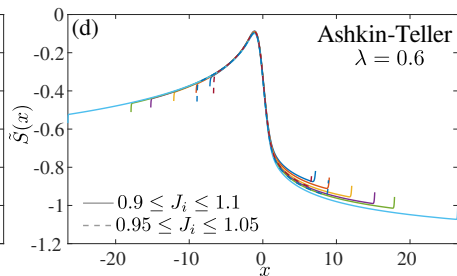
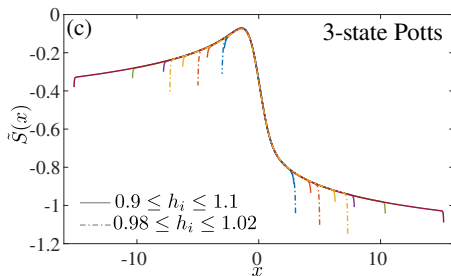
$$\tilde{S}(x) \sim S(i) + \frac{c}{12} \log \delta - aC_i$$



NC, Phys. Rev. B 110, 144401 (2024)

Entanglement entropy. 3-state Potts and Ashkin-Teller

$$\tilde{S}(x) \sim S(i) + \frac{c}{12} \log \delta - aC_i$$



NC, Phys. Rev. B 110, 144401 (2024)

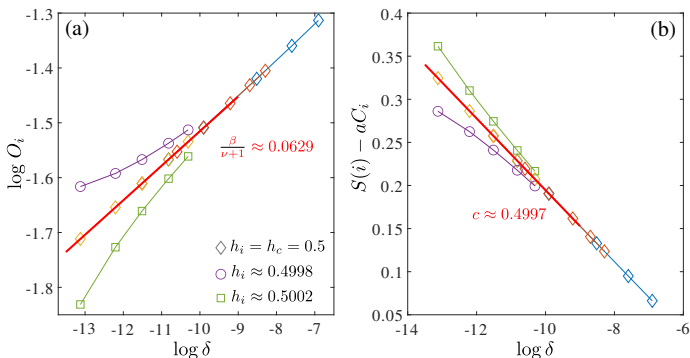
What if...

- the location of the critical point
- the set of critical exponents
- the central charge

are not known beforehand?

Ratio $\frac{\beta}{1+\nu}$ and central charge at $g = g_c$

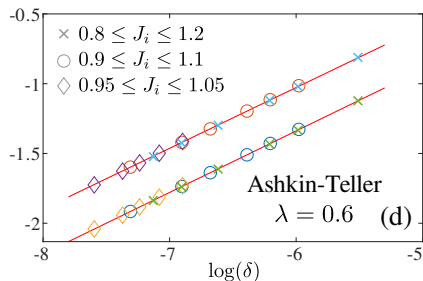
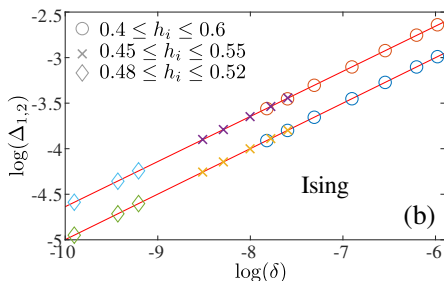
$$O_i|_{h_i=h_c} \propto \delta^{\frac{\beta}{1+\nu}} \quad \text{and} \quad S_i - aC_i|_{h_i=h_c} \propto \frac{c}{12} \log \delta$$



- Theory: $\frac{\beta}{1+\nu} = 1/16 = 0.0625$ and $c = 0.5$
- **Separatrix = critical point**

Critical exponent ν from the energy gap

$$\Delta \propto \xi^{-1} \Rightarrow \Delta \propto \delta^{\frac{\nu}{1+\nu}}$$



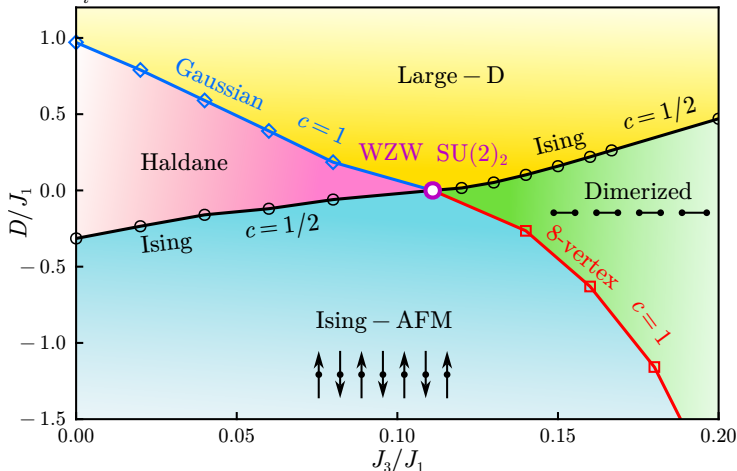
- Ising $\frac{\nu}{1+\nu} = 0.5$. Numerics: 0.4995
- Ashkin-Teller ($\lambda = 0.6$) $\frac{\nu}{1+\nu} \approx 0.4366$. Numerics: 0.4411 (within 1%)
- The minimal gap = the critical gap (fixed BC is probably a key)

Deconfined quantum criticality

Quantum phase transition between two ordered phases

New challenge: boundary conditions

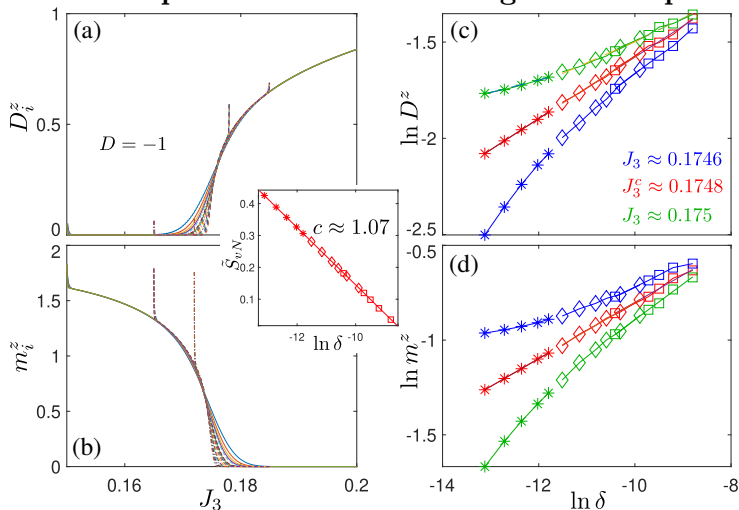
$$H = \sum_i J_1 \mathbf{S}_i \cdot \mathbf{S}_{i+1} + J_{3i} [(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{S}_{i+2}) + \text{h.c.}] + D(S_i^z)^2,$$



Pronk, La Riviere, NC, Phys. Rev. B 111, L220412 (2025)

Scan-DMRG easily realizes two different boundary conditions

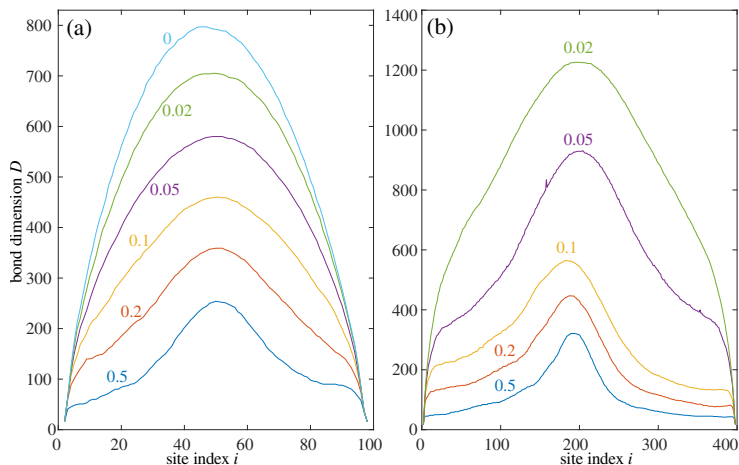
Two order parameters... and a single critical point!



Pronk, La Riviere, NC, Phys. Rev. B 111, L220412 (2025)

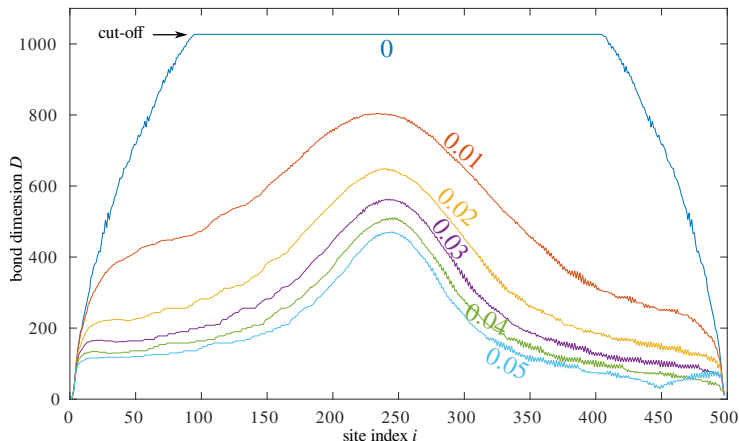
Computational advantage of scan-DMRG

Reduced bond dimension. Example 1: Ashkin-Teller model



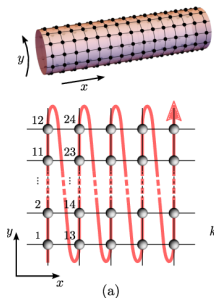
even small gradients lead to a great complexity reduction

Reduced bond dimension. Example 2: $J_1 - J_2 - J_3$



even small gradients lead to a great complexity reduction

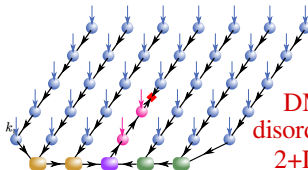
2D-DMRG



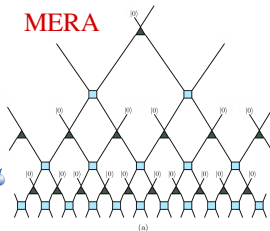
(a)

Y-DMRG

Comb / Fork DMRG
Tree tensor networks



MERA

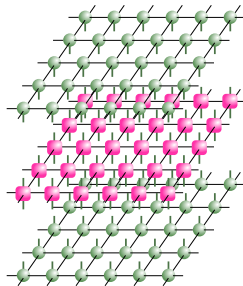


(a)

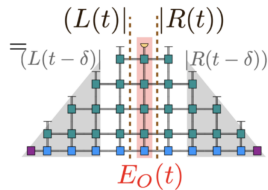


DMRG-X
disorder-DMRG
2+D iPEPS

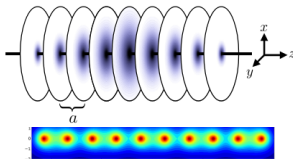
2D iPEPS



TEBD/TDVP/Light-cone TN



Sliced DMRG
Gausslets





PhD and postdoc positions available

nchepiga.github.io/homepage

Deadlines:

18 Nov 2025 (postdoc)

8 Jan 2026 (PhD)

Questions?

n.chepiga@tudelft.nl